Europ. J. Combinatorics (1982) **3**, 85–94

Enumeration of Partitions with Hooklengths

BRUCE SAGAN

0. INTRODUCTION AND PRELIMINARIES

Partitions of an integer have found extensive application in combinatorics [1, 14, 17], group representation theory [6, 13, 15], and the theory of algorithms [10, 12]. The component parts of a partition can be arranged linearly, in the plane or even associated with the elements of an arbitrary partially ordered set. One of the important properties of linear partitions is that their generating function can be written

$$\prod_{i=1}^{n} \frac{1}{1-X^{i}}, \tag{0.1}$$

where n is the number of parts. Hence we might ask what other types of partitions have a generating function expressable as a finite product with factors $1/(1-X^h)$, $h \in \{1, 2, 3, \ldots\}$. To date, three such families of partitions are known: reverse plane partitions, shifted reverse plane partitions and rooted trees. These partitions are said to have hooklengths referring to the exponents h found in the generating function.

Richard Stanley [16] first proved that reverse plane partitions and rooted trees have hooklengths. More recently, Emden Ganser disposed of the shifted case [7]. These proofs involve clever manipulation of generating functions; however, it was hoped that a more combinatorial demonstration could be given. In [11], Hillman and Grassl provided an algorithmic and, in some sense, natural proof for reverse plane partitions. This paper extends the Hillman–Grassl technique to cover the two remaining families and several realted partitions as well.

Let us make the ideas presented above more precise. Consider a *partially ordered set* (poset), $\mathscr{P}$, having n elements. A *reverse $\mathscr{P}$-partition* is an order preserving map $\rho : \mathscr{P} \to \{0, 1, 2, \ldots\}$, i.e. $x \leq y$ in $\mathscr{P}$ implies $\rho(x) \leq \rho(y)$. We say that ρ is *a reverse $\mathscr{P}$-partition of m* if $\sum_{x\in\mathscr{P}} \rho(x) = m$. The unique reverse $\mathscr{P}$-partition of zero will be denoted θ.

The *generating function* for reverse $\mathscr{P}$-partitions is given by

$$F(\mathscr{P}, X) = \sum_{m=0}^{\infty} a_m X^m, \tag{0.2}$$

where a_m is the number of reverse $\mathscr{P}$-partitions of m. The poset $\mathscr{P}$ is said to have *hooklengths* if there are a finite number of positive integers h_i ($i = 1, 2, \ldots, n = |\mathscr{P}|$) such that the generating function (0.2) can be expressed as a product

$$\sum_{m=0}^{\infty} a_m X^m = \prod_{i=1}^{n} \frac{1}{1-X^{h_i}}. \tag{0.3}$$

Next we consider three examples of posets having hooklengths.

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a *linear partition* of the integer n if $\lambda_1, \lambda_2, \ldots, \lambda_r$ are integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ and $\sum_i \lambda_i = n$. Consider an array of n cells or nodes into r left-justified rows with λ_i cells in row i. This array, $\mathscr{S}$, is called the *shape* or *Ferrers diagram* of λ (see Figure 1).

Research supported in part by the National Science Foundation.

0195–6698/82/010085+10 \$02.00/0 © 1982 Academic Press Inc. (London) Limited

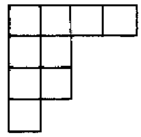


FIGURE 1. The shape of (4, 2, 2, 1).

Denote by (i, j) the cell in the ith row and jth column of the diagram. The partial order

$$(i, j) \leqslant (i', j') \quad \text{iff } i \leqslant i' \quad \text{and} \quad j \leqslant j' \tag{0.4}$$

turns $\mathscr{S}$ into a poset. In this context a reverse $\mathscr{S}$-partition is called a *reverse plane partition* of shape λ. We will let R stand for a reverse plane partition and denote the part of R in cell (i, j) by r_{ij}. Figure 2 gives an example of such a partition.

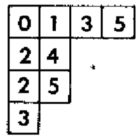


FIGURE 2. A reverse plane partition of shape (4, 2, 2, 1).

The *hook* of node (i, j), written H_{ij}, includes the node iteslef and all cells directly to the right or below, i.e.

$$H_{ij} = \{(i, j') | j' \geqslant j\} \cup \{(i', j) | i' \geqslant i\}. \tag{0.5}$$

The hooks of the (1, 3) and (2, 1) cells for the shape (4, 2, 2, 1) are displayed in Figure 3. Finally, the *hooklengths* for $\mathscr{S}$ are given by $h_{ij} = |H_{ij}|$, e.g. in Figure 3, $h_{13} = 2$ and $h_{21} = 4$.

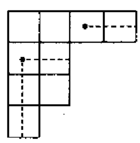


FIGURE 3. Two hooks of (4, 2 2 1).

A linear partition $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_r^*)$ is *strict* if $\lambda_1^* > \lambda_2^* > \cdots > \lambda_r^* \geqslant 0$. The *shifted shape*, $\mathscr{S}^*$, of λ^* is an array of $\sum_i \lambda_i^*$ cells in r rows with row i containing λ_i^* nodes and intended $i-1$ spaces as in Figure 4. Again (i, j) denotes the cell in the ith row and jth column (so $i \leqslant j$) and the cells are given the partial order (0.4). A *shifted reverse plane partition*, R^*, with parts r_{ij}^* is defined in the obvious way.

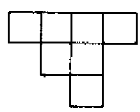


FIGURE 4. The shifted shape (4, 2, 1).

As for the *shifted hook* of cell (i, j), let

$$H_{ij}^* = \{(i, j') | j' \geqslant j\} \cup \{(i', j) | i' \geqslant i\} \cup \{(j+1, j') | j' \geqslant j+1\}. \tag{0.6}$$

If the last set in this union is empty, the hook is said to be of *type* 1 and is like a non-shifted hook, cf. (0.5). Otherwise the hook is of *type* 2. For $\lambda^* = (4, 2, 1)$, the hooks H^*_{11} (of type 2) and H^*_{13} (of type 1) are given in Figure 5. The associated *shifted hooklengths* are $h^*_{11} = 6$ and $h^*_{13} = 4$.

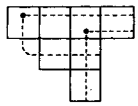
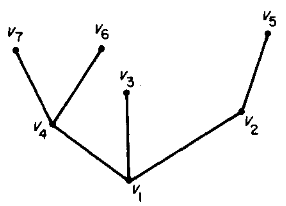


FIGURE 5. Two hooks of $\lambda^* = (4, 2, 1)$.

The last family of posets with hooklengths is the collection of rooted trees. A poset $\mathscr{T}$ is a *rooted tree* if it has a unique minimal element and the Hasse diagram of $\mathscr{T}$ is a tree in the graph-theoretic sense of the term. T will stand for any given reverse $\mathscr{T}$-partition. If v is a node of $\mathscr{T}$ then the hook of v is

$$H^\dagger_v = \{w \in \mathscr{T} | w \geqslant v\} \tag{0.7}$$

with corresponding hooklengths $h^\dagger_v = |H^\dagger_v|$. Figure 6 displays an example of rooted tree and its hooklengths.



$h^\dagger_{v1} = 7, h^\dagger_{v2} = 2, \quad h^\dagger_{v4} = 3$

$h^\dagger_{v3} = h^\dagger_{v5} = h^\dagger_{v6} = h^\dagger_{v6} = h^\dagger_{v7} = 1$

FIGURE 6. A rooted tree with hooklengths.

1. The Algorithm

We must now show that the three families introduced above are indeed hooklength posets, i.e. that their generating functions are of the desired form (0.3). The Hillman–Grassl algorithm accomplishes this for reverse plane partitions. By generalizing their proof to an arbitrary hooklength poset, $\mathscr{P}$, we can apply the same procedure to the other two families.

In equation (0.3), the coefficient of X^m on the left-hand side is the number of reverse $\mathscr{P}$-partitions of m. The corresponding coefficient on the right counts multisets ("sets" whose members may be repeated) of nodes

$$M = \{\{v_1, v_2, \ldots, v_k\}\} \quad \text{such that } \sum_{i=1}^{k} h_{v_i} = m. \tag{1.1}$$

Hence we must find a bijective correspondence between reverse $\mathscr{P}$-partitions of m and multisets of nodes whose hooklengths sum to m.

Intuitively, this bijection may be constructed in the following manner. Given a reverse $\mathscr{P}$-partition $\rho = \rho_1$ of m, we strip off a hooklength by subtracting one from h_{v_1} parts of ρ_1 for some v_1. This yields a new partition ρ_2 of $m - h_{v_1}$ from which we can strip off another hooklength h_{v_2}. We can clearly iterate this process until left with the trivial partition θ of zero. Thus ρ can be decomposed into hooklengths satisfying (1.1), where the nodes $v_1, v_2, \ldots, v_k$ will be determined at each stage by the form of the $\mathscr{P}$-partition.

To show that the above procedure is bijective, we must be able to reverse it, i.e. start with a multiset of hooklengths and reconstruct the partition ρ from which they came.

To do this must know in what order the hooklengths were removed (the pivot ordering given below). Now take the last vertex v_k whose hooklength was removed and add one back to h_{v_k} nodes of θ to obtain ρ_k. Then use the penultimate hooklength $h_{v_{k-1}}$ to reconstitute ρ_{k-1}, and so on until we have $\rho_1 = \rho$ back again.

To describe the order in which hooklengths are removed from ρ, we introduce a total order Π on the nodes of $\mathcal{P}$. This order need not be consistent with the partial order in $\mathcal{P}$ so, to avoid confusion, when referring to the total order we will call v_i a *pivot* and denote it by π_{v_i} or simply π_i. When we substract the hooklengths from a reverse $\mathcal{P}$-partition, it will turn out that the corresponding pivots will form a non-decreasing sequence in the order Π. Hence (1.1) becomes

$$\pi_1 \leqslant \pi_2 \leqslant \cdots \leqslant \pi_k \quad \text{with} \sum_{i=1}^{k} h_{\pi_i} = m. \tag{1.2}$$

We now give a formal description of the algorithm. Starting with a reverse $\mathcal{P}$-partition ρ of m we derive a sequence of pivots using procedure H.

H1. Set $\rho_1 \leftarrow \rho$, $i \leftarrow 1$.

H2. If $\rho_i = \theta$, terminate.

H3. Subtract 1 from h_{π_i} parts of ρ_i (π_i will be determined by ρ_i and Π) to obtain a new reverse $\mathcal{P}$-partition ρ_{i+1}.

H4. Set $i \leftarrow i+1$ and return to H2.

To reconstruct the partition $\rho = \rho_1$ from (1.1), we first order the nodes with respect to the pivot ordering to obtain (1.2). Now we apply the inverse algorithm G.

G1. Set $\rho_{k+1} \leftarrow \theta$, $i \leftarrow k$.

G2. If $i = 0$, terminate.

G3. Add 1 to h_{π_i} parts of ρ_{i+1} to obtain ρ_i.

G4. Set $i \leftarrow i-1$ and return to G2.

With these preliminaries, Hillman and Grassl's result can be stated:

THEOREM 1. *Fix a shape $\mathcal{S}$ with hooks given by* (0.5) *and give the nodes of $\mathcal{S}$ the total order*

$$(i, j) < (i', j') \quad \textit{iff } i < i' \textit{ or } i = i' \textit{ and } j > j'.$$

Then H1–H4 *and* G1–G4 *define a bijection between reverse $\mathcal{S}$-partitions of m and non-decreasing sequences of nodes whose hooklengths sum to m (where the subtraction and addition rules are given by* (3.2) *and* (4.1)). *Hence*

$$F(\mathcal{S}, X) = \prod_{(i,j)\in\mathcal{S}} \frac{1}{1 - X^{h_{ij}}}.$$

Next we consider reverse $\mathcal{T}$-partitions where $\mathcal{T}$ is a rooted tree.

2. ROOTED TREES

In this setting the details are particularly simple. Let Π be any linear extension of the partial order $\mathcal{T}$. Now given a reverse $\mathcal{T}$-partition, $T = T_1$, pick the smallest π_{v_1} with non-zero part. Subtract 1 from the $h^{\dagger}_{v_1}$ nodes in $H^{\dagger}_{v_1}$. Since $\mathcal{T}$ is a tree, the result, T_2, is still a reverse $\mathcal{T}$-partition and the procedure can be iterated k times until there are no non-zero parts left, i.e. $T_{k+1} = \theta$. To reverse everything merely add 1 back to the nodes in $H^{\dagger}_{v_k}, H^{\dagger}_{v_{k-1}}, \ldots, H^{\dagger}_{v_1}$ in turn where $\pi_k \geqslant \pi_{k-1} \geqslant \cdots \geqslant \pi_1$. Thus, we have the following theorem.

THEOREM 2. *Rooted trees have hooklengths, i.e.*

$$F(\mathscr{T}, X) = \prod_{v \in \mathscr{T}} \frac{1}{1 - X^{h_v^{\dagger}}}.$$

An example is carried out in Figure 7.

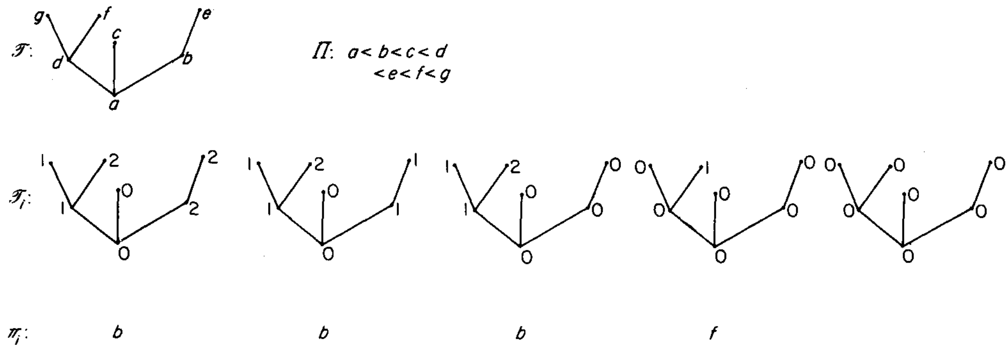


FIGURE 7. The Hillman–Grassl algorithm applied to trees (for H read left to right, for G read right to left).

3. SHIFTED REVERSE PLANE PARTITIONS AND PATHS

We now adapt H1–H4 to the shifted case.

THEOREM 3. *Given a fixed shifted shape $\mathscr{S}^*$, then the generating function for reverse $\mathscr{S}^*$-partitions is*

$$\prod_{(i,j) \in \mathscr{S}^*} \frac{1}{1 - X^{h_{ij}^*}}.$$

PROOF. Impose a total order on the cells of $\mathscr{S}^*$ given by

$$(i, j) < (i', j') \quad \text{iff } i < i' \text{ or } i = i' \text{ and } j > j'. \tag{3.1}$$

Let $R^* = R_1^*$ be a shifted reverse $\mathscr{S}^*$-partition and let (a, b) be the right-most, highest node of R_1^* with non-zero part. A path p_1^* of nodes from which ones will be subtracted is defined inductively:

$$(a, b) \in p_1^* \quad \text{and if } (i, j) \in p_1^*, \quad \text{then} \quad \begin{cases} (i, j-1) \in p_1^* & \text{if } r_{ij-1}^* = r_{ij}^* \\ (i+1, j) \in p_1^* & \text{otherwise,} \end{cases} \tag{3.2}$$

i.e. move down unless forced to move left so as not to violate the non-decreasing condition along the rows once the ones are subtracted. Continue this process until the induction rule fails. At this point the path must be at the lower end of some column, say column c. Two situations can now occur.

(1) $c \geqslant r$ (where r is the number of rows of $\mathscr{S}^*$) in which case p_1^* terminates and the ones are subtracted. We have taken away a total of h_{ac}^* giving a type 1 pivot $\pi_1 = (a, c)$. This is essentially the same as the path constructed in the original Hillman–Grassl algorithm for non-shifted reverse plane partitions. See, for example, the path in Figure 8.

(2) $c < r$. Now ones are subtracted along p_1^* and the path continues by:

$$(c+1, c+1) \in p_1^* \quad \text{and if } (i, j) \in p_1^*, \quad \text{then} \quad \begin{cases} (i-1, j) \in p_1^* & \text{if } r_{i-1j}^* = r_{ij}^* \\ (i, j+1) \in p_1^* & \text{otherwise,} \end{cases}$$

i.e. move right unless forced to move up (r_{ij}^* refers to the (i, j) part *after* the subtraction).

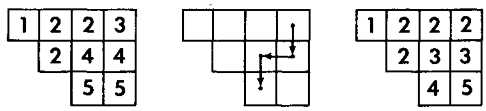


FIGURE 8. A type 1 path (or return path).

Note that this second half of p_1^* lies strictly below the first. Indeed after subtraction along the first half each part on that portion is strictly less than its neighbor below and hence cannot be reached by the second half. Hence p_1^* will terminate when it reaches the right end of some row, say row d. Now subtract ones along the second half of p_1^*. This time we have taken away a total of h_{ad-1}^* giving a type 2 pivot $\pi_1=(a, d-1)$. Figure 9 illustrates this case.

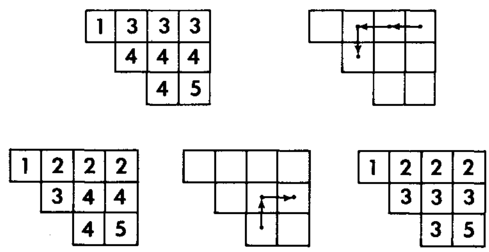


FIGURE 9. A type 2 path (or return path).

The reader can easily verify in either case that the new array R_2^* is still a reverse $\mathscr{S}^*$-partition. Iterating this process we get a sequence of reverse $\mathscr{S}^*$-partitions R_1^*, $R_2^*, \ldots, R_{k+1}^*=\theta$ and corresponding pivots $\pi_1, \pi_2, \ldots, \pi_k$. We also have the following lemma.

LEMMA 4. *The sequence $\pi_1, \pi_2, \ldots, \pi_k$ is non-decreasing.*

PROOF. It suffices to show that $\pi_1 \leqslant \pi_2$. If $\pi_1=(u, v)$ and $\pi_2=(w, x)$, then, by the way the initial nodes of p_1^* and p_2^* are chosen, $w \geqslant u$. If $w>u$, we are done. Considering $w=u$, there are three possibilities. When p_1^* and p_2^* are both type 1 paths then $x \leqslant v$ because no node of p_2^* can lie to the right of p_1^* (for suppose that $(i, j) \in p_1^*$ and $(i, j-1) \in p_1^*$, thus $r_{ij}^*=r_{ij-1}^*$ in both R_1^* and R_2^*, so if we have $(i, j) \in p_2^*$ then $(i, j-1) \in p_2^*$). If p_1^* is type 1 and p_2^* type 2, then $x<r \leqslant v$. Finally if p_1^* is type 2, this forces p_2^* to be type 2: as before no node of the first half of p_2^* can lie to the right of the first half of p_1^*. Hence no node of the second half of p_2^* can lie below the second half of p_1^*, again yielding $x \leqslant v$. Thus in all cases $\pi_1 \leqslant \pi_2$.

This completes the verification of (1.2) and shows that H1–H4 are well defined.

4. THE INVERSE ALGORITHM G

Now we must construct a sequence of reverse $\mathscr{S}^*$-partitions $\theta=R_{k+1}^*, R_k^*, \ldots, R_1^*$ from a pivot sequence $\pi_k \geqslant \pi_{k-1} \geqslant \cdots \geqslant \pi_1$. Assuming R_{l+1}^* $(1 \leqslant l \leqslant k)$ has been built, consider $\pi_l=(u, v)$. If π_l is of type 1 let e be the lowest row in column v. Construct the return path q_l^* as follows:

$$(e, v) \in q_l^* \quad \text{and if } (i, j) \in q_l^*, \quad \text{then} \quad \begin{cases}(i, j+1) \in q_l^* & \text{if } r_{ij+1}^*=r_{ij}^* \\ (i-1, j) \in q_l^* & \text{otherwise.}\end{cases} \tag{4.1}$$

Note that "*otherwise*" includes the possibility that r^*_{ij+1} is undefined because $(i, j+1) \notin \mathcal{S}^*$. The return path terminates when it reaches the right end of row u and then 1 is added to each node in q^*_l.

If π_l is of type 2, let f be the right-most column in row $v+1$. Define the first half of q^*_l by

$$(v+1, f) \in q^*_l \quad \text{and if } (i, j) \in q^*_l, \quad \text{then} \quad \begin{cases} (i+1, j) \in q^*_l & \text{if } r^*_{i+1j} = r^*_{ij} \\ (i, j-1) \in q^*_l & \text{otherwise} \end{cases}$$

until q^*_l reaches a node (g, g). After adding ones along this part of q^*_l, continue the return path using the induction rule for type 1 pivots and initial node $(g-1, g-1)$. The second half terminates as with type 1 pivots, when it reaches the right end of row u and then 1 is added to each node in this half. By construction the new array will still be a reverse $\mathcal{S}^*$-partition. For examples of return paths read Figures 8 and 9 from right to left.

It is not clear *a priori* that q^*_l ever reaches the end of row u. The total order (3.1) is essential to our proof.

LEMMA 5. $(u, u+\lambda^*_u - 1) \in q^*_l$.

PROOF. Consider the return path q^*_{l+1} for the pivot $\pi_{l+1} = (w, x) \geq (u, v)$. If $w > u$, then we have $r^*_{iu} = 0$ for all i hence q^*_l will be forced to the right end of row u by the first alternative in (4.1) (this argument also works when $l = k$ even though q^*_{k+1} and π_{k+1} do not exist). If $w = u$, then $x \leq v$ and there are three possibilities: π_l and π_{l+1} are both type 1, π_l is type 1 and π_{l+1} type 2, or π_l is type 2 which forces π_{l+1} to be type 2. The proofs that q^*_l reaches $(u, u+\lambda^*_u)$ are similar in all three cases so I will only present the last.

q^*_l starts in row $v+1$ on or below q^*_{l+1} which begins in row $x+1$. In fact no node of the first half of q^*_l can lie above the first half of q^*_{l+1} as we will show. For if q^*_l reaches $(i, j) \in q^*_{l+1}$ and $(i+1, j) \in q^*_{l+1}$, then $r^*_{i+1j} = r^*_{ij}$ (both before and after addition of ones), so $(i+1, j) \in q^*_l$. If the first portion of q^*_{l+1} terminates at (c, c) and q^*_l ends at (d, d), it follows that $c \leq d$. Hence, the second half of q^*_l starts to the right of or on the second half of q^*_{l+1}. By similar reasoning, every node of this half of q^*_l lies to the right of or on the second half of q^*_{l+1}, and since q^*_{l+1} reaches the node $(u, u+\lambda^*_u)$ (by assumption) so must q^*_l. This proves Lemma 5 showing that the inverse algorithm G1–G4 is well defined. This also finishes the proof of Theorem 3.

At this point an example is in order. Figure 10 illustrates the Hillman–Grassl algorithm applied to a shifted reverse (4, 2, 1)-partition.

	1222	1111	0000	0000	0000	0000
	34	23	12	11	00	00
R^*_i:	4	3	3	2	1	0

π_i: (1, 1) (1, 1) (2, 3) (2, 2) (3, 3)

FIGURE 10. The Hillman–Grassl algorithm applied to shifted shapes (for H read left to right, for G read right to left).

5. OTHER GENERATING FUNCTIONS

There are many generating functions in the theory of partitions which can be written in the form

$$F(X) = \prod_i \frac{1}{1-X^{h_i}} \quad \text{or} \quad F(X) = \prod_i \frac{1-X^{g_i}}{1-X^{h_i}}, \tag{5.1}$$

where the h_i and g_i take on positive integral values as i ranges over some index set [5, 8, 9]. Some of these can be attacked by the method outlined in Section 1.

THEOREM 6 (MACMAHON [14]). *The generating function for all plane partitions (shape not fixed) is*

$$\prod_{i=1}^{\infty} \frac{1}{(1-X^i)^i}. \tag{5.2}$$

(*A plane partition is an assignment of positive integers to a non-shifted shape which is non-increasing along rows and columns.*)

PROOF. Consider the "infinite shape" defined by $\mathscr{S}=\{(i,j) \mid i,j \geq 1\}$. Let the hook of the (i,j) cell be

$$H_{ij}=\{(i,j') \mid j' \leq j\} \cup \{i',j) \mid i' \leq i\} \tag{5.3}$$

with hooklength $h_{ij}=|H_{ij}|=i+j-1$. Note that with these definitions the generating function (5.2) becomes

$$\prod_{(i,j)\in\mathscr{S}} \frac{1}{1-X^{h_{ij}}}.$$

Now, given a plane partition M with parts m_{ij}, imbed it in the infinite shape using the map

$$\mu:\mathscr{S}\to\mathbb{N} \text{ defined by } \mu(i,j)=\begin{cases} m_{ij} & \text{if } (i,j)\in M \\ 0 & \text{if } (i,j)\notin M. \end{cases}$$

Define a total order Π on $\mathscr{S}$ by

$$(i,j)<(i',j') \quad \text{iff } j>j' \text{ or } j=j' \text{ and } i<i'. \tag{5.4}$$

The path p in M starts at the right-most highest non-zero node of $\mathscr{S}$ and continues left unless forced to move down, terminating when it reaches the left-hand edge. The details of the proof are similar to those of Sections 3 and 4 and are left to the reader.

Similarly we obtain various refinements of Theorem 6.

THEOREM 7. *The coefficient of X^m in*

$$\prod_{i=1}^{\infty} \frac{1}{(1-X^i)^{\min(i,s)}} \tag{5.5}$$

is the number of plane partitions of M with at most s rows (*MacMahon* [14]).

The coefficient of X^mY^d in

$$\prod_{i=1}^{\infty} \frac{1}{(1-X^iY)^i} \tag{5.6}$$

is the number of plane partitions of m whose diagonal elements sum to d, i.e., $\sum_i m_{ii}=d$ (*Stanley* [18]).

The coefficient of X^mY^d in

$$\prod_{i=1}^{\infty} \frac{1}{(1-X^iY)^{\min(i,s)}} \tag{5.7}$$

is the number of plane partitions of m with at most s row whose diagonal elements sum to d (*Stanley* [18]).

PROOF. Equation (5.5) follows from consideration of the shape $\mathcal{S} = \{(i,j)|1 \leq i \leq s, j \geq 1\}$ with hooks given by (5.3). To derive (5.6) and (5.7), merely note that each path intersects the diagonal of $\mathcal{S}$ exactly once.

Now let us consider a generating function of the type (5.1) with a non-trivial numerator.

THEOREM 8. (SYLVESTER [19]). *The generating function for linear partitions into at most s parts with each part of size at most t is*

$$\prod_{i=1}^{s} \frac{1-X^{t+i}}{1-X^i}. \tag{5.8}$$

PROOF. Call the partitions counted in this theorem *st-partitions*. Letting $a(m)$ be the number of st-partitions of m, we wish to show that

$$\sum_{m=0}^{\infty} a(m)X^m = \prod_{i=1}^{s} \frac{1-X^{t+i}}{1-X^i}$$

or

$$\frac{\sum_m a(m)X^m}{\prod_i (1-X^{t+i})} = \prod_{i=1}^{s} \frac{1}{1-X^i}. \tag{5.9}$$

The right-hand side of (5.9) counts partitions with at most s parts and no restriction on part size. Thus it suffices to find a bijection between partitions of m' into at most s parts and pairs consisting of an st-partition of m and a multiset of the integers $t+1$, $t+2, \ldots, t+s$ summing to m'' such that $m' = m + m''$. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $r \leq s$, if $\lambda_1 \leq t$, then map λ to itself paired with the empty multiset. If $\lambda_1 > t$, consider the plane partition M_1 of shape λ all of whose entries are 1. Subtract ones along the path p_1 through M_1 defined by

$$(1, \lambda_1) \in p_1 \text{ and if } (i,j) \in p_1 \quad \text{then} \quad \begin{cases} (i+1, j) \in p_1 & \text{if } m_{i+1j} = m_{ij} = 1 \\ (i, j-1) \in p_1 & \text{otherwise.} \end{cases}$$

Terminate p_1 at the lowest node of M_1 in column $\lambda_1 - t$, i.e. at $(i, \lambda_1 - t)$ where $(i+1, \lambda_1 - t) \notin M_1$. ($p_1$ will pass through this node since it always stays on the boundary of M_1). Note that at this point we will have subtracted 1 from $t+i$ nodes, $1 \leq i \leq s$.

Using this procedure repeatedly we obtain a sequence of plane partitions M_1, $M_2, \ldots, M_{k+1}$ (where M_{k+1} is the plane partition of an st-partition) and associated integers $t+i_1, t+i_2, \ldots, t+i_k$. Also the $t+i_l$ are in non-decreasing order. For if the paths p_l and p_{l+1} terminate in columns j_l and j_{l+1} respectively, then $j_l > j_{l+1}$ and so $i_l \leq i_{l+1}$. The construction of the inverse map is done in the obvious manner.

As an example let $\lambda = (11, 6, 2)$, $s = 3$ and $t = 2$ (see Figure 11).

$$\begin{array}{llllll}
 & 11111111111 & 11111111 & 11111 & 1111 & 1 \\
N_i: & 111111 & 111111 & 11111 & 11 & 1 \\
 & 11 & 11 & 11 & 11 & 1 \\
 & & & & & \\
t+h_i: & 2+1 & 2+2 & 2+2 & 2+3 &
\end{array}$$

i.e. $(11, 6, 2) \leftrightarrow ((1, 1, 1), \{\{3, 4, 4, 5\}\})$

FIGURE 11. The bijection of Theorem 8.

Theorem 8 can be put in the context of the rest of this paper as follows. Using the shape $\mathcal{S}=\{(i, 1)|1\leq i\leq s\}$ and hooks as defined by (5.3), we can write (5.8) as

$$\sum_m a(m)X^m = \prod_{(i,j)\in\mathcal{S}} \frac{1-X^{h_{ij}+t}}{1-X^{h_{ij}}}.$$

The integers $h_{ij}+t$ are called *modified hooklengths* and (5.4) induces a total order on modified pivots (definition clear) which in this case is just the usual ordering of integers as used in the theorem. This may all seem needlessly pedantic but the notation becomes more more suggestive when considering rst-partitions, i.e. plane partitions with at most r parts in each row, at most s parts in each column and part size at most t.

THEOREM 9 (MACMAHON [14]). *The generating function for rst-partitions is*

$$\prod_{(i,j)\in\mathcal{S}} \frac{1-X^{h_{ij}+t}}{1-X^{h_{ij}}},$$

where $\mathcal{S}=\{(i,j)|1\leq i\leq s, 1\leq j\leq r\}$ *and hooklengths are as in Theorem* 6.

To date, no purely bijective proof of Theorem 9 is known [2, 3, 4]. Is it possible to extend the methods presented here to this result?

REFERENCES

1. G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, Mass., 1976.
2. G. E. Andrews, Implications of the MacMahon conjecture, in *Combinatoire et Représentation du Groupe Symétrique, Strasbourg, 1976* (D. Foata, ed.), Lecture Notes in Mathematics No. 579, Springer-Verlag, Berlin, 1977, pp. 287–296.
3. G. E. Andrews, Plane partitions (I): the MacMahon conjecture, *Adv. in Math.* (to appear).
4. G. E. Andrews, Plane partitions (II): the equivalence of the Bender–Knuth and MacMahon conjectures (to appear).
5. E.A. Bender and D. E. Knuth, Enumeration of plane partitions, *J. Combin. Theory Ser. A* **13** (1972), 40–54.
6. J. S. Frame, G. de B. Robinson and R. M. Thrall, The hook lengths of $\mathcal{G}_n$, *Canad. J. Math.* **6** (1954),
7. E. Gansner, Matrix Correspondences and the enumeration of plane partitions, Ph.D. Thesis, M.I.T., 1978.
8. B. Gordon and L. Houten, Notes on plane partitions I, II, *J. Combin. Theory* **4** (1968), 72–80, 81–99.
9. B. Gordon and L. Houten, Notes on plane partitions III, *Duke Math. J.* **36** (1969), 801–829.
10. C. Greene, A. Nijenhuis and H. S. Wilf, A probabilistic proof of a formula for the number of young tableaux of a given shape, preprint.
11. A. P. Hillman, and R. M. Grassl, Reverse plane partitions and tableau hook numbers, *J. Combin. Theory Ser. A* **21** (1976), 216–221.
12. D. E. Knuth, *The Art of Computer Programming*, vol. 3, Addison-Wesley, Reading, Mass., 1973.
13. D. E. Littlewood, *The Theory of Group Characters*, 2nd ed., Oxford University Press, Oxford, 1950.
14. P. A. MacMahon, *Combinatory Analysis*, vols. 1 and 2, Cambridge University Press, 1915, 1916; reprinted by Chelsea, New York, 1960.
15. G. de B. Robinson, On the representations of the symmetric group, *Amer. J. Math.* **60** (1938), 745–760; **69** (1947), 286–298; **70** (1948), 227–294.
16. R. P. Stanley, Ordered structures and partitions, Thesis, Harvard University, 1971.
17. R. P. Stanley, Theory and application of plane partitions I and II, *Stud. Appl. Math.* **50** (1971), 167–188, 259–279.
18. R. P. Stanely, The conjugate trace and trace of a plane partition, *J. Combin. Theory Ser. A* **14** (1973), 53–65.
19. J. J. Sylvester, A constructive theory of partitions, arranged in three acts, an interact and as exodion, *Amer. J. Math.* **5** (1882), 251–330; **6** (1884), 334–336.

B. SAGAN
Department of Mathematics, University of Michigan,
Ann Arbor, Michigan 48109, U.S.A.