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On Selecting a Random Shifted Young Tableau

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A probabilistic algorithm of Greene, Nijenhuis, and Wilf is applied to shifted shapes. It is proved that this procedure yields a Young tableau of the given shape and that all such tableaux are equally likely.

1. INTRODUCTION AND DEFINITIONS

The study of standard and generalized Young tableaux has led to interesting results in quite a number of areas of mathematics. These arrays are of importance in the representation theory of the symmetric and general linear groups [12], in invariant theory [7], and in connection with various combinatorial problems [1]. In addition, many algorithms have been developed that manipulate the tableaux and their entries [5, 6]. We will be particularly concerned with a probabilistic procedure developed by Greene *et al.* [4]. This algorithm generates a standard Young tableau of fixed shape at random. In so doing, it also provides a proof of the hook formula (Eq. (1.1) below) which enumerates such tableaux.

There is another family of arrays, the shifted Young tableaux, that exhibit many similarities to their unshifted cousins [8]. It is the purpose of this paper to show that the Greene, Nijenhuis, and Wilf procedure can be extended to shifted tableaux. Interestingly enough, the algorithms for both types of tableaux are identical, but the proof that all tableaux are equally likely is much more difficult in the shifted case.

Let us now make the concepts introduced in the last two paragraphs more precise. A *partition* of the integer n is a vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ having integral components such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and $\Sigma_i \lambda_i = n$. The *shape* of λ, S, is an array of n cells or nodes into r left-justified rows with λ_i cells in row i. Finally by placing the numbers $1, 2, \ldots, n$ into the cells of S so that the rows and columns increase we obtain a *standard Young Tableau* of shape S (see Fig. 1). The number of standard Young tableaux of shape S will be denoted by f_S.

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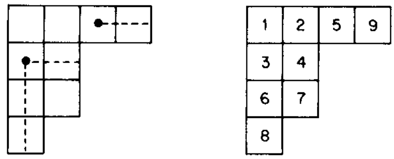


FIG. 1. The shape of (4, 2, 2, 1) with two of its hooks, and a standard Young tableau of that shape.

Let (i, j) be the cell in the ith row and jth column of S. The *hook* of node (i, j) is defined by

$$H_{ij} = \{(i, j') | j' \geq j\} \cup \{(i', j) | i' \geq i\}$$

while the *hooklength* is $h_{ij} = |H_{ij}|$. In Fig. 1 we have used dotted lines to indicate the (1, 3) and (2, 1) hooks with corresponding hooklengths $h_{13} = 2$, $h_{21} = 4$. Given a fixed shape S it is possible to express the number of standard Young tableaux of that shape in terms of the hooklengths:

$$f_{\mathrm{S}} = \frac{n!}{\prod_{(i,j) \in \mathrm{S}} h_{ij}}. \tag{1.1}$$

For the original proof of this result the reader can consult [2].

A partition $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_r^*)$ is *strict* if $\lambda_1^* > \lambda_2^* > \cdots > \lambda_r^* > 0$. The *shifted shape*, S^*, of λ^* is an array of $\Sigma_i \lambda_i^*$ cells into r rows with row i containing λ_i^* cells and indented $i - 1$ spaces. Placing the integers from 1 to n in S^* so that the rows and columns increase yields a *shifted standard Young tableau* (see Fig. 2). We will let f_{S}^* be the number of shifted standard Young tableaux of a given shifted shape.

The *shifted hook* of cell $(i, j) \in \mathrm{S}^*$ is, by definition,

$$H_{ij}^* = \{(i, j') | j' \geq j\} \cup \{(i', j) | i' \geq i\} \cup \{(j + 1, j') | j' \geq j + 1\}$$

with *shifted hooklength* $h_{ij}^* = |H_{ij}^*|$. In Fig. 2, two of the hooklengths are $h_{11}^* = 6$ and $h_{13}^* = 4$. Furthermore, the hook formula (1.1) has an analog:

$$f_{\mathrm{S}}^* = \frac{n!}{\prod_{(i,j) \in \mathrm{S}^*} h_{ij}^*}. \tag{1.2}$$

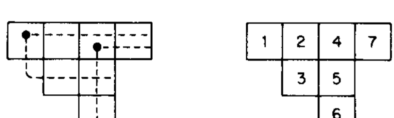


FIG. 2. The shifted shape (4, 2, 1) with two of its hooks, and a shifted standard Young tableau of that shape.

In the next section we use this formula to develop an algorithm for picking a tableau of given shape at random. Although all our results are stated for shifted shapes, everything that we say in Sections 2 and 3 can also be applied to the unshifted case (as is done in [4]).

2. The Algorithm

It is convenient to have a notation for the nodes (i, j) of a shape without reference to coordinates. We will use the letters v and w to stand for cells of a shifted shape S^* (thus $v = (i, j)$ for some coordinates i and j). We will also use the notation H_v for H_{ij}, h_v for h_{ij}, etc.

Consider a fixed shape S^* with n nodes. Suppose we have an algorithm which produces a standard shifted Young tableau of shape S^* at random. In fact, suppose this procedure gives each tableau of shape S^* with probability $\Pi_{v\in S^*}h_v/n!$. The hook formula (1.2) follows immediately, for

$$\text{(number of tableaux)} \cdot \text{(probability of each tableau)} = 1$$

or $f_S^* \cdot \Pi h_v^*/n! = 1$; thus $f_S^* = n!/\Pi h_v^*$.

Our candidate for the desired algorithm is

GNW1. Set $i \leftarrow 1$.

GNW2. (Now S^* has $n - i + 1$ nodes.) Set $j \leftarrow 1$ and pick a node $v_1 \in S^*$ with probability $1/(n - i + 1)$.

GNW3. If $h_{v_j}^* > 1$, pick a node $v_{j+1} \in H_{v_j}^* - \{v_j\}$ with probability $1/(h_{v_j}^* - 1)$. If not, go to GNW5.

GNW4. Set $j \leftarrow j + 1$ and return to GNW3.

GNW5. (Now $h_{v_j}^* = 1$.) Give node v_j the label $n - i + 1$ and delete v_j from S^*.

GNW6. Set $i \leftarrow i + 1$. If $i \leq n$, return to GNW2; if $i > n$, terminate.

The sequence of nodes $v_1, v_2, v_3, \ldots$ generated by one pass through the outer loop of this algorithm is called a *trial.* A trial must end after a finite number of steps, for if $v_{j+1} \in H_{v_j}^* - \{v_j\}$ then clearly $h_{v_{j+1}}^* < h_{v_j}^*$. Thus the $h_{v_j}^*$ form a decreasing sequence of positive integers which terminates when $h_{v_l} = 1$ for some l. Of course, as each node of S^* is labeled and deleted in GNW5, the hooks and hooklengths for the next loop must be modified appropriately (the h_v will stay the same or decrease by one). Hence the algorithm will have labeled every node of S^* after n trials.

Theorem 1. *Given a shifted shape* S^*, *the algorithm* GNW1–6 *will produce a standard shifted Young tableau of shape* S^* *at random. In addition, the probability of obtaining any particular tableau is* $\Pi_{v\in S^*}h_v^*/n!$.

Proof (beginning). If v_l is the terminal node of our first trial, then v_l must be maximal in S^*; i.e., there is no node of S^* below or to the left of v_l. GNW5 gives v_l the label n and removes it from further consideration. It follows by induction on n that the algorithm does indeed produce a standard labeling of S^* where the rows and columns increase.

We must now prove that all tableaux are equally likely with probability $\prod h_v^*/n!$. Let w be any maximal node of S^*. By induction on n, it suffices to show that the probability of terminating the first trial at w is

$$\operatorname{prob}(w) = \frac{\prod_{v \in S^*} h_v^*/n!}{\prod_{v \in S_w^*} h_v^*/(n-1)!},$$

where S_w^* is the shape S^* with w deleted and the hooklengths in the denominator are taken in S_w^*. Letting $W = \{v \in S^* | w \in H_v \text{ and } w \neq v\}$, we can write this fraction as

$$\operatorname{prob}(w) = \frac{1}{n} \prod_{v \in W} \frac{h_v^*}{h_v^* - 1}$$

where all hook lengths are now taken in S^*, or

$$\operatorname{prob}(w) = \frac{1}{n} \prod_{v \in W} \left(1 + \frac{1}{h_v^* - 1}\right). \tag{2.1}$$

Suppose that w has coordinates (α, β). Then the elements of W are of three types: those in the αth row, those in the βth column, and those in the $(\alpha - 1)$st column. See Fig. 3. Letting $a_i = h_{i\beta}^* - 1$, $b_j = h_{\alpha j}^* - 1$, and $c_i = h_{i\alpha-1}^* - 1$ we can rewrite Eq. (2.1) as

$$\operatorname{prob}(w) = \operatorname{prob}(\alpha, \beta) = \prod_{i=1}^{\alpha-1} \left(1 + \frac{1}{a_i}\right) \prod_{j=\alpha}^{\beta-1} \left(1 + \frac{1}{b_j}\right) \prod_{i=1}^{\alpha-1} \left(1 + \frac{1}{c_i}\right), \tag{2.2}$$

where by convention empty products are equal to 1.

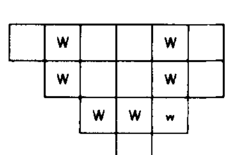


FIG. 3. $w = (3, 5)$ and $W = \{(3, 3), (3, 4), (1, 5), (2, 5), (1, 2), (2, 2)\}$.

We will need a readable notation for the terms in the product expansion of (2.2). Given subsets $A \subseteq \{a_1, a_2, \ldots, a_{\alpha-1}\}$, $B \subseteq \{b_\alpha, b_{\alpha+1}, \ldots, b_{\beta-1}\}$, and $C \subseteq \{c_1, c_2, \ldots, c_{\alpha-1}\}$, where $A = \{a_{i_1}, a_{i_2}, \ldots\}$, $B = \{b_{j_1}, b_{j_2}, \ldots\}$, and $C = \{c_{k_1}, c_{k_2}, \ldots\}$, we define

$$\frac{1}{A \cdot B \cdot C} = \frac{1}{a_{i_1}a_{i_2} \ldots b_{j_1}b_{j_2} \ldots c_{k_1}c_{k_2} \ldots}$$

(empty sets contribute a factor of 1 to the denominator). In the sections that follow we will give a combinatorial interpretation to the sums of such terms.

3. Terms of the Form $1/(A \cdot B)$, $(C = \varnothing)$

If $v = (i, j)$ is a node of S^* we consider I_v = the *horizontal projection of* $v = i$ and J_v = the *vertical projection of* $v = j$. The reason for this redundant notation is to prevent indexing confusion later on in the proof. Now given a trial $v_1, v_2, \ldots, v_l = (\alpha, \beta)$ the *horizontal projection of the trial* is the set $I = \{i \mid i = I_{v_k}$ for some k, $1 \le k \le l$, and $i \neq \alpha\}$. Similarly the *vertical projection of the trial* is $J = \{j \mid j = J_{v_k}$ for some k, $1 \le k \le l$, and $j \neq \beta\}$. Let $\text{prob}(\alpha\beta \mid IJ)$ denote the probability of terminating at (α, β) using a trial with horizontal projection I and vertical projection J.

Proposition 2. *Let (α, β), $I = \{i_1, i_2, \ldots\}$, and $J = \{j_1, j_2, \ldots\}$ be given. Further suppose that $1 \le i_k < \alpha$ and $\alpha \le j_k < \beta$ for all k. Then*

$$\text{prob}(\alpha\beta \mid IJ) = \frac{1}{A \cdot B},$$

where $A = \{a_{i_1}, a_{i_2}, \ldots\}$ and $B = \{b_{j_1}, b_{j_2}, \ldots\}$.

For variety's sake we present an example before the proof. Consider the shifted shape in Fig. 3 with $w = (\alpha, \beta) = (3, 5)$. Let us pick $I = \{1\}$ and $J = \{3, 4\}$. Then the possible trials for prob $(3, 5 \mid \{1\}, \{3, 4\})$ are just

$$v_1 = (1, 3);\ v_2 = (1, 4);\ v_3 = (1, 5);\ v_4 = (3, 5),$$

$$v_1 = (1, 3);\ v_2 = (1, 4);\ v_3 = (3, 4);\ v_4 = (3, 5),$$

and

$$v_1 = (1, 3);\ v_2 = (3, 3);\ v_3 = (3, 4);\ v_4 = (3, 5)$$

with corresponding probabilities $\frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{3}$, $\frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{2}$, and $\frac{1}{6} \cdot \frac{1}{3} \cdot \frac{1}{2}$ so

$$\text{prob}(3, 5 \mid \{1\}, \{3, 4\}) = \tfrac{1}{6} \cdot \tfrac{1}{5} \cdot \tfrac{1}{3} + \tfrac{1}{6} \cdot \tfrac{1}{5} \cdot \tfrac{1}{2} + \tfrac{1}{6} \cdot \tfrac{1}{3} \cdot \tfrac{1}{2} = \tfrac{1}{18}.$$

On the other hand $A = \{a_1\}$ and $B = \{b_3, b_4\}$, where $a_1 = h^*_{15} - 1 = 3$, $b_3 = h^*_{33} - 1 = 3$, and $b_4 = h^*_{34} - 1 = 2$, so

$$\frac{1}{A \cdot B} = \frac{1}{3 \cdot 3 \cdot 2} = \frac{1}{18}.$$

Proof of Proposition 2. This proof is essentially the one given by Greene, Nijenhuis, and Wilf but we include it for completeness.

Case 1. Suppose that either I or J is empty. For definiteness let $I = \varnothing$ (the case $J = \varnothing$ being similar). Then the only possible trial is

$$v_1 = (\alpha, j_1), \qquad v_2 = (\alpha, j_2), \ldots, v_l = (\alpha, \beta),$$

which instantly gives

$$\operatorname{prob}(\alpha\beta | IJ) = \left(\frac{1}{b_{j_1}}\right)\left(\frac{1}{b_{j_2}}\right) \cdots = \frac{1}{B} = \frac{1}{A \cdot B},$$

where $A = \varnothing$.

Case 2. Now we induct on $m = |I \cup J|$. Note that $m = 0$ is taken care of in Case 1. In fact we can assume that $I \neq \varnothing$, $J \neq \varnothing$ and that the proposition has been proved for $m - 1$. Since $I = \{i_1, i_2, \ldots\} \neq \varnothing$ and $J = \{j_1, j_2, \ldots\} \neq \varnothing$ it follows that $v_1 = (i_1, j_1)$, where $i_1 \neq \alpha$ and $j_1 \neq \beta$. Furthermore there are only two possible choices for v_2, i.e., $v_2 = (i_2, j_1)$ or $v_2 = (i_1, j_2)$. Hence if we let $I_1 = I - \{i_1\}$ and $J_1 = J - \{j_1\}$ then we can apply the induction hypothesis yielding

$$\begin{aligned}
\operatorname{prob}(\alpha\beta | IJ) &= \frac{1}{h^*_{i_1 j_1} - 1}\left[\operatorname{prob}(\alpha\beta | I_1 J) + \operatorname{prob}(\alpha\beta | I J_1)\right] \\
&= \frac{1}{h^*_{i_1 j_1} - 1}\left[\frac{1}{\hat{a}_{i_1} a_{i_2} \cdots b_{j_1} b_{j_2} \cdots} + \frac{1}{a_{i_1} a_{i_2} \cdots \hat{b}_{j_1} b_{j_2} \cdots}\right] \\
&= \frac{a_{i_1} + b_{j_1}}{h^*_{i_1 j_1} - 1}\left(\frac{1}{a_{i_1} a_{i_2} \cdots b_{j_1} b_{j_2} \cdots}\right) \qquad (3.1)
\end{aligned}$$

(where $\hat{}$ indicates that the factor is deleted). But it is easy to see that

$$h^*_{i_1 j_1} - 1 = \left(h^*_{i_1 \beta} - 1\right) + \left(h^*_{\alpha j_1} - 1\right) = a_{i_1} + b_{j_1} \qquad (3.2)$$

so (3.1) reduces to

$$\operatorname{prob}(\alpha\beta | IJ) = \frac{1}{A \cdot B} \text{ as desired. } \square$$

Let $(\mathfrak{a}, \mathfrak{b})$ be any node of S^* such that $\mathfrak{a} \leq \alpha$, $\mathfrak{b} \leq \beta$ and let $\operatorname{prob}(\alpha\beta | \mathfrak{a}\mathfrak{b})$ be the probability of terminating a trial at (α, β) given that the trial starts at $v_1 = (\mathfrak{a}, \mathfrak{b})$. Considering those $(\mathfrak{a}, \mathfrak{b})$ where $1 \le \mathfrak{a} \le \alpha$ and $\alpha \le \mathfrak{b} \le \beta$ we have

$$\operatorname{prob}(\alpha\beta|\mathfrak{a}\mathfrak{b}) = \sum_{I,J} \operatorname{prob}(\alpha\beta|IJ),$$

where the sum is over all $I \subseteq \{\mathfrak{a}, \mathfrak{a}+1, \ldots, \alpha-1\}$ and $J \subseteq \{\mathfrak{b}, \mathfrak{b}+1, \ldots, \beta-1\}$ such that $\mathfrak{a} \in I$ and $\mathfrak{b} \in J$. If $A \subseteq \{a_{\mathfrak{a}}, a_{\mathfrak{a}+1}, \ldots, a_{\alpha-1}\}$ and $B \subseteq \{b_{\mathfrak{b}}, b_{\mathfrak{b}+1}, \ldots, b_{\beta-1}\}$ have index sets I and J, respectively, then the above sum becomes, via Proposition 2,

$$\begin{aligned}\operatorname{prob}(\alpha\beta|\mathfrak{a}\mathfrak{b}) &= \sum_{A,B} \frac{1}{A \cdot B} \\ &= \frac{1}{a_{\mathfrak{a}} b_{\mathfrak{b}}} \cdot \prod_{\mathfrak{a}<i<\alpha} \left(1 + \frac{1}{a_i}\right) \prod_{\mathfrak{b}<j<\beta} \left(1 + \frac{1}{b_j}\right),\end{aligned}$$

where, again, empty products are equal to one. We immediately have

PROPOSITION 3. *Consider those nodes* $(\mathfrak{a}, \mathfrak{b})$ *of* S^* *satisfying* $1 \le \mathfrak{a} \le \alpha$ *and* $\alpha \le \mathfrak{b} \le \beta$; *then*

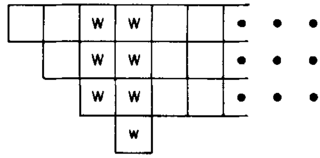
$$\sum_{\substack{(\mathfrak{a},\mathfrak{b}) \\ 1 \le \mathfrak{a} \le \alpha \\ \alpha \le \mathfrak{b} \le \beta}} \operatorname{prob}(\alpha\beta|\mathfrak{a}\mathfrak{b}) = \prod_{i=1}^{\alpha-1} \left(1 + \frac{1}{a_i}\right) \prod_{j=1}^{\beta-1} \left(1 + \frac{1}{b_j}\right). \qquad \square$$

Up to this point our proof has not differed significantly from that given for the unshifted case. However, we will need new techniques to account for the trials starting from nodes $(\mathfrak{a}, \mathfrak{b})$ where $1 \le \mathfrak{a}, \mathfrak{b} < \alpha$.

4. A Counterexample and a Step Forward

Knowing Proposition 2 we might hope at this point to express each of the terms $1/(A \cdot B \cdot C)$, $C \neq \varnothing$, as the probability of a certain type of trial, i.e., as the sum of probabilities of all trials of a given type. However, this cannot be done, as we will see in the following example.

Consider the shape of the strict partition $\lambda^* = (1001, 101, 11, 1)$ with $w = (\alpha, \beta) = (4, 4)$, part of which is shown in Fig. 4. This is by no means the smallest counterexample but our choice of λ^* ensures that the hook-lengths in each row of the portion under consideration will differ by an order of magnitude. This will make the number of trials we need to check very small. Since $W = \{(1, 4), (2, 4), (3, 4), (1, 3), (2, 3), (3, 3)\}$ we have $a_1 = 1000$, $a_2 = 100$, $a_3 = 10$, $c_1 = 1001$, $c_2 = 101$, $c_3 = 11$ (there are no



Part of the shape of $\lambda^* = (1001, 101, 11, 1)$ with $w = (4, 4)$

(1, 1);	(1, 2);	(1, 3);	(1, 4);	(4, 4)		
(1, 1);	(1, 2);	(1, 3);	(1, 4);	(2, 4);	(4, 4)	
(1, 1);	(1, 2);	(1, 3);	(1, 4);	(3, 4);	(4, 4)	
(1, 1);	(1, 2);	(1, 3);	(1, 4);	(2, 4);	(3, 4);	(4, 4)
(1, 1);	(1, 2);	(1, 3);	(2, 3);	(2, 4);	(4, 4)	
(1, 1);	(1, 2);	(1, 3);	(2, 3);	(2, 4);	(3, 4);	(4, 4)
(1, 1);	(1, 2);	(1, 3);	(2, 3);	(3, 3);	(3, 4);	(4, 4)
(1, 1);	(1, 2);	(1, 3);	(2, 3);	(3, 3);	(4, 4)	
(1, 1);	(1, 2);	(1, 4);	(2, 4);	(3, 4);	(4, 4)	
(1, 1);	(1, 2);	(2, 2);	(2, 3);	(2, 4);	(4, 4)	
(1, 1);	(1, 2);	(2, 2);	(2, 3);	(2, 4);	(3, 4);	(4, 4)
(1, 1);	(1, 2);	(2, 2);	(2, 3);	(3, 3);	(3, 4);	(4, 4)

FIG. 4. Trials with probability less than or equal to $1/a_1a_2a_3c_1c_2c_3$.

b's). Now consider the term

$$\frac{1}{a_1a_2a_3c_1c_2c_3} = \frac{1}{(1000)(100)(10)(1001)(101)(11)}$$

$$\approx 0.89919 \times 10^{-12}$$

In Fig. 4 we have listed all trials that have probability less than or equal to $1/a_1a_2a_3c_1c_2c_3$. It is straightforward, but tedious, to verify that there is no subset of the listed trials whose probabilities sum to

$$\frac{1}{1000 \cdot 100 \cdot 10 \cdot 1001 \cdot 101 \cdot 11}.$$

To overcome this difficulty we need to look at *sums* of the terms $1/(A \cdot B \cdot C)$, but first let us simplify the problem by focusing our attention on a special type of trial. A trial $v_1, v_2, \ldots, v_l = (\alpha, \beta)$ is called a *basic trial* if $J_{v_k} < \alpha$ or $J_{v_k} = \beta$ for all k, $1 \leq k \leq l$. In other words a basic trial never contains a node in columns α through $\beta - 1$ (which are precisely those columns corresponding to the b's). The probability of reaching (α, β) from $(\mathfrak{a}, \mathfrak{b})$ using only basic trials is denoted $\mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,\mathfrak{a}\mathfrak{b})$. If $\alpha \leq \mathfrak{a} < \beta$ then we have automatically that $\mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,\mathfrak{a}\mathfrak{b}) = 0$. In addition if (i, j) is *any* node in S^* let $\mathrm{prob}_{\mathrm{P}}(ij\,|\,\mathfrak{a}\mathfrak{b})$ be the probability of reaching (i, j) from $(\mathfrak{a}, \mathfrak{b})$ by a *partial trial* that contains only nodes v_k such that $J_{v_k} < \alpha$, with the possible exception of (i, j) itself; i.e., we are permitted to have $j \geq \alpha$. Note that for all i we have

$$\mathrm{prob}_{\mathrm{P}}(i\alpha|\mathfrak{a}\mathfrak{b}) = \mathrm{prob}_{\mathrm{P}}(i\alpha + 1|\mathfrak{a}\mathfrak{b}) = \cdots = \mathrm{prob}_{\mathrm{P}}(i\beta|\mathfrak{a}\mathfrak{b}).$$

LEMMA 4. *If* $\mathfrak{a}$, $\mathfrak{b} < \alpha$ *then*

$$\mathrm{prob}(\alpha\beta|\mathfrak{a}\mathfrak{b}) = \mathrm{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}\mathfrak{b}) \cdot \prod_{\alpha \leq j < \beta} \left(1 + \frac{1}{b_j}\right).$$

Proof. Since every trial from $(\mathfrak{a}, \mathfrak{b})$ to (α, β) must have a first node (i, j) with $\alpha \leq j \leq \beta$ (and of course $1 \leq i \leq \alpha$)

$$\begin{aligned}
\mathrm{prob}(\alpha\beta|\mathfrak{a}\mathfrak{b}) &= \sum_{\substack{(i,j)\\ 1 \leq i \leq \alpha \\ \alpha \leq j \leq \beta}} \mathrm{prob}_{\mathrm{P}}(ij|\mathfrak{a}\mathfrak{b}) \cdot \mathrm{prob}(\alpha\beta|ij) \\
&= \sum_{1 \leq i \leq \alpha} \sum_{\alpha \leq j \leq \beta} \mathrm{prob}_{\mathrm{P}}(ij|\mathfrak{a}\mathfrak{b}) \cdot \mathrm{prob}(\alpha\beta|ij) \\
&= \sum_{1 \leq i \leq \alpha} \sum_{\alpha \leq j \leq \beta} \mathrm{prob}_{\mathrm{P}}(i\beta|\mathfrak{a}\mathfrak{b}) \cdot \mathrm{prob}(\alpha\beta|ij) \\
&= \sum_{1 \leq i \leq \alpha} \mathrm{prob}_{\mathrm{P}}(i\beta|\mathfrak{a}\mathfrak{b}) \cdot \sum_{\alpha \leq j \leq \beta} \mathrm{prob}(\alpha\beta|ij) \\
&= \sum_{1 \leq i \leq \alpha} \mathrm{prob}_{\mathrm{P}}(i\beta|\mathfrak{a}\mathfrak{b}) \\
&\quad \cdot \left[\frac{1}{a_i} \cdot \prod_{i < k < \alpha} \left(1 + \frac{1}{a_k}\right) \cdot \prod_{\alpha \leq j < \beta} \left(1 + \frac{1}{b_j}\right)\right]
\end{aligned}$$

(this follows from Proposition 2, empty products being replaced by one and a_i being replaced by one if $i = \alpha$)

$$\begin{aligned}
&= \left[\sum_{1 \leq i \leq \alpha} \mathrm{prob}_{\mathrm{P}}(i\beta|\mathfrak{a}\mathfrak{b}) \cdot \frac{1}{a_i} \cdot \prod_{i < k < \alpha} \left(1 + \frac{1}{a_k}\right)\right] \cdot \prod_{\alpha \leq j < \beta} \left(1 + \frac{1}{b_j}\right) \\
&= \mathrm{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}\mathfrak{b}) \cdot \prod_{\alpha \leq j < \beta} \left(1 + \frac{1}{b_j}\right). \qquad \square
\end{aligned}$$

In the light of Lemma 4 and Proposition 3 we can finish the proof of Theorem 1 by showing that the basic trials account for the terms of (2.2) having only factors of $1/a_i$ and/or $1/c_i$. In other words we must prove the following.

Proposition 5. *Let (α, β) be a maximal node of* S^*. *Then we have*

$$\sum_{(\mathfrak{a}, \mathfrak{b})} \mathrm{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}\mathfrak{b}) = \prod_{i=1}^{\alpha-1}\left(1+\frac{1}{a_i}\right)\left(1+\frac{1}{c_i}\right). \tag{4.1}$$

The next section is devoted to proving this proposition.

5. Enumeration of Basic Trials

We need a few preliminary definitions. Consider $A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\} \subseteq \{a_1, a_2, \ldots, a_{\alpha-1}\}$, $1 < i_1 < i_2 < \cdots < i_r$, and $C = \{c_{j_1}, c_{j_2}, \ldots, c_{j_s}\} \subseteq \{c_1, c_2, \ldots, c_{\alpha-1}\}$, $1 < j_1 < j_2 < \cdots < j_s$, so that

$$\frac{1}{A \cdot C} = \frac{1}{a_{i_1}a_{i_2} \ldots a_{i_r}c_{j_1}c_{j_2} \ldots c_{j_s}}$$

as usual. Now define four operators on this product (N (for none), R (for right), L (for left) and T (for together)) by

$$\left(\frac{1}{A \cdot C}\right)_{\mathrm{N}} = \frac{1}{a_{i_1-1}a_{i_2-1} \cdots a_{i_r-1}c_{j_1-1}c_{j_2-1} \cdots c_{j_s-1}},$$

$$\left(\frac{1}{A \cdot C}\right)_{\mathrm{R}} = \frac{1}{a_{i_1-1}a_{i_2-1} \cdots a_{i_r-1}a_{\alpha-1}c_{j_1-1}c_{j_2-1} \cdots c_{j_s-1}},$$

$$\left(\frac{1}{A \cdot C}\right)_{\mathrm{L}} = \frac{1}{a_{j_1-1}a_{j_2-1} \cdots a_{j_s-1}c_{i_1-1}c_{i_2-1} \cdots c_{i_r-1}c_{\alpha-1}},$$

and

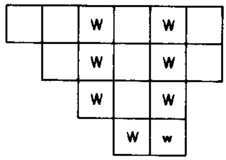
$$\left(\frac{1}{A \cdot C}\right)_{\mathrm{T}} = \frac{1}{a_{j_1-1}a_{j_2-1} \cdots a_{j_s-1}a_{\alpha-1}c_{i_1-1}c_{i_2-1} \cdots c_{i_r-1}c_{\alpha-1}}.$$

If either A or C is empty they merely contribute 1 to the bottom to the denominator, e.g.,

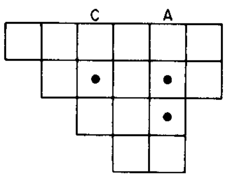
$$\left(\frac{1}{c_{j_1}c_{j_2}}\right)_{\mathrm{R}} == \frac{1}{a_{\alpha-1}c_{j_1-1}c_{j_2-1}}.$$

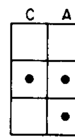
Now extend each of these operators to sums by linearity, for example,

$$\left(\sum_i \frac{1}{A_iC_i}\right)_{\mathrm{N}} = \sum_i \left(\frac{1}{A_iC_i}\right)_{\mathrm{N}}.$$

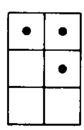


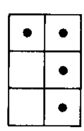
The shape of $\lambda^* = (6, 5, 3, 2)$ with $w = (4, 5)$

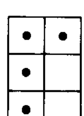
$\frac{1}{a_2a_3c_2} =$ 

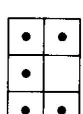
or abbreviated 

A particular product

$\left(\frac{1}{a_2a_3c_2}\right)_N = \frac{1}{a_1a_2c_1} =$ 

$\left(\frac{1}{a_2a_3c_2}\right)_R = \frac{1}{a_1a_2a_3c_1} =$ 

$\left(\frac{1}{a_2a_3c_2}\right)_L = \frac{1}{a_1c_1c_2c_3} =$ 

$\left(\frac{1}{a_2a_3c_2}\right)_T = \frac{1}{a_1a_3c_1c_2c_3} =$ 

$$\left(\frac{1}{a_2a_3c_2}\right)_U = \frac{1}{a_1a_2c_1} + \frac{1}{a_1a_2a_3c_1} + \frac{1}{a_1c_1c_2c_3} + \frac{1}{a_1a_3c_1c_2c_3}$$

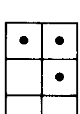
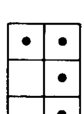
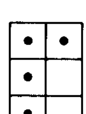
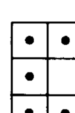
$=$  $+$  $+$  $+$ 

FIG. 5. The operators.

Finally define the *upward operator*, U, by

$$\left(\sum_i \frac{1}{A_iC_i}\right)_{\mathrm{U}} = \left(\sum_i \frac{1}{A_iC_i}\right)_{\mathrm{N}} + \left(\sum_i \frac{1}{A_iC_i}\right)_{\mathrm{R}} + \left(\sum_i \frac{1}{A_iC_i}\right)_{\mathrm{L}} + \left(\sum_i \frac{1}{A_iC_i}\right)_{\mathrm{T}}.$$

It is helpful to view these operations in diagram form. For example, consider the shape of $\lambda^* = (6, 5, 3, 2)$ with $w = (\alpha, \beta) = (4, 5)$. Figure 5 illustrates the product $1/a_2a_3c_2$ in this shape and shows the effects of each of our operators upon it.

LEMMA 6. *Given any row* $\mathfrak{a}$ *with* $1 \le \mathfrak{a} < \alpha$; *then if*

$$\sum_{\mathfrak{b}=\mathfrak{a}+1}^{\beta} \mathrm{prob_B}(\alpha\beta\,|\,\mathfrak{a}+1\mathfrak{b}) = \sum_i \frac{1}{A_iC_i}$$

it follows that

$$\sum_{\mathfrak{b}=\mathfrak{a}}^{\beta} \mathrm{prob_B}(\alpha\beta\,|\,\mathfrak{a}\mathfrak{b}) = \sum_i \left(\frac{1}{A_iC_i}\right)_{\mathrm{U}}. \tag{5.1}$$

In fact we have explicitly

$$\sum_{\mathfrak{b}=\mathfrak{a}}^{\beta} \mathrm{prob_B}(\alpha\beta\,|\,\mathfrak{a}\mathfrak{b}) = \left(\frac{1}{a_{\mathfrak{a}}} + \frac{1}{c_{\mathfrak{a}}} + \frac{1}{a_{\mathfrak{a}}c_{\mathfrak{a}}}\right) \cdot \prod_{\mathfrak{a}<i<\alpha}\left(1 + \frac{1}{a_i}\right)\left(1 + \frac{1}{c_i}\right). \tag{5.2}$$

In words, (5.2) says that the basic trials starting in row $\mathfrak{a}$ account for those terms of (2.2) containing only a_i's and c_i's where $\mathfrak{a} \le i < \alpha$ and at least one of $1/a_{\mathfrak{a}}$, $1/c_{\mathfrak{a}}$ as a factor. This lemma immediately implies Proposition 5, and thereby completes the proof of Theorem 1, for all terms of (5.2) are contained in (4.1). Furthermore take any term $1/(A \cdot C) \neq 1$ in (4.1), where $A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\}$, $C = \{c_{j_1}, c_{j_2}, \ldots, c_{j_s}\}$, and let $\mathfrak{a}_0 = \min\{i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_s\}$. Then Lemma 6 assures us that $1/(A \cdot C)$ occurs in $\mathrm{prob_B}(\alpha\beta\,|\,\mathfrak{a}_0\mathfrak{b})$ for some $\mathfrak{b}$ but not in the probabilities for basic trials from any other row. Hence we are done except for the proof of the lemma.

Proof of Lemma 6. We will actually prove a stronger result determining implicitly the form of $\mathrm{prob_B}(\alpha\beta\,|\,\mathfrak{a}\mathfrak{b})$ for all $(\mathfrak{a}, \mathfrak{b}) \in \mathbf{S}^*$. Of course $\mathrm{prob_B}(\alpha\beta\,|\,\mathfrak{a}\mathfrak{b}) = 0$ if $\alpha \le \mathfrak{b} < \beta$ or $\mathfrak{b} > \beta$. There are four other cases.

(1) If

$$\mathrm{prob_B}(\alpha\beta\,|\,\mathfrak{a}+1\beta) = \sum_i \frac{1}{A_i} \tag{5.3}$$

then

$$\operatorname{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}\beta) = \sum_i \left(\frac{1}{A_i}\right)_{\mathrm{N}} + \left(\frac{1}{A_i}\right)_{\mathrm{R}}. \tag{5.4}$$

In addition $1/a_{\mathfrak{a}}$ is a factor of every term in (5.4).

(2) If

$$\operatorname{prob}(\alpha\beta|\mathfrak{a}+1\alpha-1) = \sum_i \frac{1}{A_i'C_i'} \tag{5.5}$$

and the A_i are as in (5.3) then we have

$$\operatorname{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}\alpha-1) = \sum_i \left(\frac{1}{A_i}\right)_{\mathrm{L}} + \left(\frac{1}{A_i}\right)_{\mathrm{T}} + \left(\frac{1}{A_i'C_i'}\right)_{\mathrm{N}} + \sum_{A_i'\neq\varnothing} \left(\frac{1}{A_i'C_i'}\right)_{\mathrm{R}}. \tag{5.6}$$

In addition $1/c_{\mathfrak{a}}$ is a factor of every term in (5.6).

(3) If the A_i', C_i' are as in (5.5) and the terms having $A_i' = \varnothing$ are denoted $1/C_i'$ then

$$\operatorname{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}\alpha-2) = \sum_i \left(\frac{1}{C_i'}\right)_{\mathrm{R}} + \left(\frac{1}{A_i'C_i'}\right)_{\mathrm{L}} + \left(\frac{1}{A_i'C_i'}\right)_{\mathrm{T}}. \tag{5.7}$$

In addition $1/a_{\mathfrak{a}}c_{\alpha-1}$ or $1/a_{\alpha-1}c_{\mathfrak{a}}$ is a factor of every term in (5.7).

(4) If $\mathfrak{b} < \alpha - 2$ and

$$\operatorname{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}+1\,\mathfrak{b}+1) = \sum_i \frac{1}{A_i''C_i''} \tag{5.8}$$

then

$$\operatorname{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}\mathfrak{b}) = \sum_i \left(\frac{1}{A_i''C_i''}\right)_{\mathrm{U}}. \tag{5.9}$$

In addition $1/a_{\mathfrak{a}}c_{\mathfrak{b}+1}$ or $1/a_{\mathfrak{b}+1}c_{\mathfrak{a}}$ is a factor of every term in (5.9).

To illustrate (1)–(4) we have put the diagrams corresponding to the terms in the sum $\operatorname{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}\mathfrak{b})$ in the $(\mathfrak{a}, \mathfrak{b})$ cell of $S^* = (6, 5, 3, 2)$, where $(\alpha, \beta) = (4, 5)$. (See Fig. 6.) We now prove each of the cases in turn.

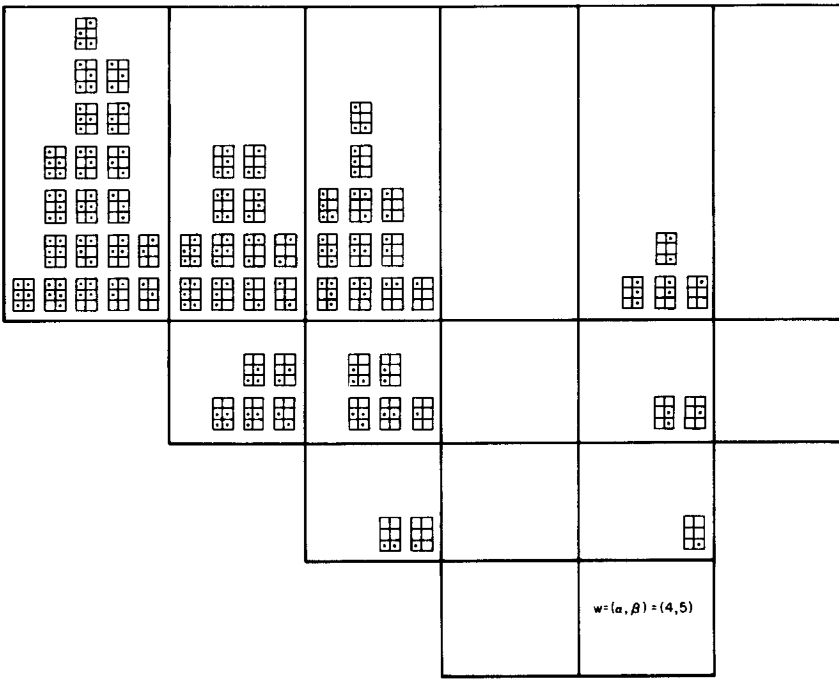


FIG. 6. The shifted shape $S^* = (6, 5, 3, 2)$ and its basic trials.

(1) We already know that

$$\mathrm{prob}_B(\alpha\beta|\mathfrak{a}\beta) = \frac{1}{a_\mathfrak{a}} \cdot \prod_{\mathfrak{a}<i<\alpha} \left(1 + \frac{1}{a_i}\right) \tag{5.10}$$

so this case follows immediately from the definitions of N and R.

(2) The trials $v_1, v_2, \ldots, v_l$ starting from $(\mathfrak{a}, \alpha - 1)$ are of two types: either $J_{v_k} = \alpha - 1$ for all $k < l$; or there is an s, $1 < s < l$, such that $J_{v_k} = \alpha - 1$ for $k < s$ and $J_{v_k} = \beta$ for $k \geq s$ (in addition $I_{v_s} = I_{v_{s-1}}$). Thus

$$\mathrm{prob}_B(\alpha\beta|\mathfrak{a}\alpha - 1) = \frac{1}{c_\mathfrak{a}} \cdot \prod_{\mathfrak{a}<i<\alpha} \left(1 + \frac{1}{c_i}\right)$$

$$+ \frac{1}{c_\mathfrak{a}} \cdot \sum_{I=\mathfrak{a}}^{\alpha-1} \left\{\frac{1}{c_I} \cdot \prod_{\mathfrak{a}<i<I} \left(1 + \frac{1}{c_i}\right)\right\} \cdot \left\{\frac{1}{a_I} \cdot \prod_{I<j<\alpha} \left(1 + \frac{1}{a_j}\right)\right\}, \tag{5.11}$$

where $1/c_I$ (in the first set of braces) is set equal to 1 when $I = \mathfrak{a}$.

Rewriting (5.11) we have

$$\begin{aligned}
\mathrm{prob}_{\mathrm{B}}(\alpha\beta \mid \mathrm{a}\alpha - 1) &= \frac{1}{c_{\mathrm{a}} c_{\alpha-1}} \cdot \prod_{\mathrm{a}<i<\alpha-1} \left(1 + \frac{1}{c_i}\right) + \frac{1}{c_{\mathrm{a}}} \cdot \prod_{\mathrm{a}<i<\alpha-1} \left(1 + \frac{1}{c_i}\right) \\
&\quad + \frac{1}{a_{\alpha-1} c_{\mathrm{a}} c_{\alpha-1}} \cdot \prod_{\mathrm{a}<i<\alpha-1} \left(1 + \frac{1}{c_i}\right) \\
&\quad + \frac{1}{c_{\mathrm{a}}} \cdot \sum_{I=\mathrm{a}}^{\alpha-2} \left\{ \frac{1}{c_I} \cdot \prod_{\mathrm{a}<i<I} \left(1 + \frac{1}{c_i}\right) \right\} \\
&\quad \cdot \left\{ \frac{1}{a_I} \cdot \prod_{I<j<\alpha-1} \left(1 + \frac{1}{a_j}\right) \right\} \\
&\quad + \frac{1}{a_{\alpha-1} c_{\mathrm{a}}} \cdot \sum_{I=\mathrm{a}}^{\alpha-2} \left\{ \frac{1}{c_I} \cdot \prod_{\mathrm{a}<i<I} \left(1 + \frac{1}{c_i}\right) \right\} \\
&\quad \cdot \left\{ \frac{1}{a_I} \cdot \prod_{I<j<\alpha-1} \left(1 + \frac{1}{a_j}\right) \right\} \\
&= \left[\frac{1}{a_{\mathrm{a}+1}} \cdot \prod_{\mathrm{a}+1<i<\alpha} \left(1 + \frac{1}{a_i}\right) \right]_{\mathrm{L}} + \left[\frac{1}{c_{\mathrm{a}+1}} \cdot \prod_{\mathrm{a}+1<i<\alpha} \left(1 + \frac{1}{c_i}\right) \right]_{\mathrm{N}} \\
&\quad + \left[\frac{1}{a_{\mathrm{a}+1}} \cdot \prod_{\mathrm{a}+1<i<\alpha} \left(1 + \frac{1}{a_i}\right) \right]_{\mathrm{T}} \\
&\quad + \left[\frac{1}{c_{\mathrm{a}+1}} \cdot \sum_{I=\mathrm{a}+1}^{\alpha-1} \left\{ \frac{1}{c_I} \cdot \prod_{\mathrm{a}+1<i<I} \left(1 + \frac{1}{c_i}\right) \right\} \right. \\
&\quad \left. \cdot \left\{ \frac{1}{a_I} \cdot \prod_{I<j<\alpha} \left(1 + \frac{1}{a_j}\right) \right\} \right]_{\mathrm{N}} \\
&\quad + \left[\frac{1}{c_{\mathrm{a}+1}} \cdot \sum_{I=\mathrm{a}+1}^{\alpha-1} \left\{ \frac{1}{c_I} \cdot \prod_{\mathrm{a}+1<i<I} \left(1 + \frac{1}{c_i}\right) \right\} \right. \\
&\quad \left. \cdot \left\{ \frac{1}{a_I} \cdot \prod_{I<j<\alpha} \left(1 + \frac{1}{a_j}\right) \right\} \right]_{\mathrm{R}}.
\end{aligned}$$

Comparison of this last expression with (5.10) and (5.11) yields (5.6).

(3) We will use reverse induction on the row a. There is an analog of Eq. (3.2) which will play an important role in what follows. Specifically, let $(i, j) \in \mathbf{S}^*$ with $i \leq j < \alpha - 1$; then

$$\begin{aligned}
h_{ij}^* - 1 &= \left(h_{i\beta}^* - 1\right) + \left(h_{j+1\alpha-1}^* - 1\right) = a_i + c_{j+1}, \\
h_{ij}^* - 1 &= \left(h_{j+1\beta}^* - 1\right) + \left(h_{i\alpha-1}^* - 1\right) = a_{j+1} + c_i.
\end{aligned} \tag{5.12}$$

To start the induction consider $\mathfrak{a} = \alpha - 2$. From the definition of a basic trial we have

$$\mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,\alpha - 2\;\alpha - 2) = \frac{1}{h^*_{\alpha-2\alpha-2} - 1}\big[\mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,\alpha - 2\;\alpha - 1)$$

$$+\mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,\alpha - 2\beta) + \mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,\alpha - 1\;\alpha - 1) + \mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,\alpha - 1\;\beta)\big].$$

Now using (5.10) and (5.11) yields

$$\mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,\alpha - 2\;\alpha - 2) = \frac{1}{h^*_{\alpha-2\,\alpha-2} - 1}$$

$$\times\left[\frac{1}{c_{\alpha-2}} + \frac{1}{c_{\alpha-2}c_{\alpha-1}} + \frac{1}{a_{\alpha-2}c_{\alpha-2}} + \frac{1}{a_{\alpha-2}a_{\alpha-1}c_{\alpha-2}}\right.$$

$$\left.+\frac{1}{a_{\alpha-1}c_{\alpha-2}c_{\alpha-1}} + \frac{1}{a_{\alpha-2}} + \frac{1}{a_{\alpha-2}a_{\alpha-1}} + \frac{1}{c_{\alpha-1}} + \frac{1}{a_{\alpha-1}c_{\alpha-1}} + \frac{1}{a_{\alpha-1}}\right]$$

$$= \frac{1}{h^*_{\alpha-2\alpha-2} - 1}\left[\frac{a_{\alpha-1} + c_{\alpha-2}}{a_{\alpha-1}c_{\alpha-2}} + \frac{a_{\alpha-2} + c_{\alpha-1}}{a_{\alpha-2}c_{\alpha-2}c_{\alpha-1}}\right.$$

$$\left.+\frac{a_{\alpha-2} + c_{\alpha-1}}{a_{\alpha-2}a_{\alpha-1}c_{\alpha-2}c_{\alpha-1}} + \frac{a_{\alpha-2} + c_{\alpha-1}}{a_{\alpha-2}c_{\alpha-1}} + \frac{a_{\alpha-2} + c_{\alpha-1}}{a_{\alpha-2}a_{\alpha-1}c_{\alpha-1}}\right]$$

$$= \frac{1}{a_{\alpha-1}c_{\alpha-2}} + \frac{1}{a_{\alpha-2}c_{\alpha-2}c_{\alpha-1}} + \frac{1}{a_{\alpha-2}a_{\alpha-1}c_{\alpha-2}c_{\alpha-1}}$$

$$+\frac{1}{a_{\alpha-2}c_{\alpha-1}} + \frac{1}{a_{\alpha-2}a_{\alpha-1}c_{\alpha-1}} \qquad \text{(by (5.12))}$$

$$= \left(\frac{1}{c_{\alpha-1}}\right)_{\mathrm{R}} + \left(\frac{1}{a_{\alpha-1}c_{\alpha-1}}\right)_{\mathrm{L}} + \left(\frac{1}{a_{\alpha-1}c_{\alpha-1}}\right)_{\mathrm{T}}$$

$$+\left(\frac{1}{c_{\alpha-1}}\right)_{\mathrm{L}} + \left(\frac{1}{c_{\alpha-1}}\right)_{\mathrm{T}},$$

which is (5.7).

Now consider the $\mathfrak{a}$th row, $\mathfrak{a} < \alpha - 2$. Clearly

$$\mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,\mathfrak{a}\alpha - 2) = \frac{1}{h^*_{\mathfrak{a}\alpha-2} - 1}\left[\sum_{\substack{(i,j)\in H^*_{\mathfrak{a}\alpha-2}\\ (i,j)\neq(\mathfrak{a},\,\alpha-2)}} \mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,ij)\right]. \quad (5.13)$$

We must show that every term of (5.7) appears in (5.13). First consider terms of the form $(1/C_i')_{\mathrm{R}}$. From (5.11) we know that

$$\left(\frac{1}{C_i'}\right)_{\mathrm{R}} = \frac{1}{a_{\alpha-1}c_{\mathfrak{a}}c_{i_2}\cdots c_{i_k}},$$

where $\mathfrak{a} < i_2 < \cdots < i_k < \alpha - 1$. Since $h^*_{\mathfrak{a}\alpha-2} - 1 = a_{\alpha-1} + c_{\mathfrak{a}}$ we have

$$\left(\frac{1}{C_i'}\right)_{\mathrm{R}} = \frac{1}{h^*_{\mathfrak{a}\alpha-2} - 1}\cdot\frac{a_{\alpha-1} + c_{\mathfrak{a}}}{a_{\alpha-1}c_{\mathfrak{a}}c_{i_2}\cdots c_{i_k}}$$

$$= \frac{1}{h^*_{\mathfrak{a}\alpha-2} - 1}\left(\frac{1}{c_{\mathfrak{a}}c_{i_2}\cdots c_{i_k}} + \frac{1}{a_{\alpha-1}c_{i_2}\cdots c_{i_k}}\right),$$

where

$$\frac{1}{c_{\mathfrak{a}}c_{i_2}\cdots c_{i_k}}$$

occurs in $\mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,\mathfrak{a}\ \alpha - 1)$ and by induction

$$\frac{1}{a_{\alpha-1}c_{i_2}\cdots c_{i_k}}$$

occurs in $\mathrm{prob}_{\mathrm{B}}(\alpha\beta\,|\,i_2\alpha - 2)$. Hence both terms will be found in (5.13) and also the $(1/C_i')_{\mathrm{R}}$ have $1/a_{\alpha-1}c_{\mathfrak{a}}$ as a factor.

Next we have terms such as $(1/A_i'C_i')_{\mathrm{L}}$. These can be split into two groups:

(i) $A_i' = \varnothing$. This case can be treated analogously to $(1/C_i')_{\mathrm{R}}$ with the roles of a and c reversed.

(ii) $A_i' \neq \varnothing$. An arithmetic lemma is required.

LEMMA 7. *Let $i_1 < i_2 < \cdots < i_s < \alpha$ be a fixed set of indices; then*

$$\sum_{k=1}^{s} \frac{1}{a_{i_1}a_{i_2}\cdots a_{i_k}c_{i_k}c_{i_{k+1}}\cdots c_{i_s}} = \sum_{k=1}^{s} \frac{1}{c_{i_1}c_{i_2}\cdots c_{i_k}a_{i_k}a_{i_{k+1}}\cdots a_{i_s}}.$$

Proof of Lemma 7. Induct on the number of indices, s, using the fact that $a_i + c_j = a_j + c_i$ for all $i, j < \alpha$. If $s = 1$ the identity is trivial. If $s > 1$

we have

$$\sum_{k=1}^{s} \frac{1}{a_{i_1}a_{i_2}\cdots a_{i_k}c_{i_k}c_{i_{k+1}}\cdots c_{i_s}}$$

$$= \frac{1}{\prod_{j=1}^{s} a_{i_j}c_{i_j}} \cdot \sum_{k=1}^{s} c_{i_1}c_{i_2}\cdots c_{i_{k-1}}a_{i_{k+1}}\cdots a_{i_s}$$

$$= \frac{1}{\prod_{j=1}^{s} a_{i_j}c_{i_j}} \left[a_{i_3}a_{i_4}\cdots a_{i_s}(a_{i_2} + c_{i_1}) + \sum_{k=3}^{s} c_{i_1}\cdots c_{i_{k-1}}a_{i_{k+1}}\cdots a_{i_s} \right]$$

$$= \frac{1}{\prod_{j=1}^{s} a_{i_j}c_{i_j}} \left[a_{i_3}a_{i_4}\cdots a_{i_s}(a_{i_1} + c_{i_2}) + \sum_{k=3}^{s} c_{i_1}\cdots \hat{c}_{i_{k-1}}a_{i_{k+1}}\cdots a_{i_s} \right]$$

$$= \frac{1}{\prod_{j=1}^{s} a_{i_j}c_{i_j}} \left[a_{i_1}a_{i_3}a_{i_4}\cdots a_{i_s} + a_{i_4}a_{i_5}\cdots a_{i_s}c_{i_2}(a_{i_3} + c_{i_1}) + \sum_{k=4}^{s} c_{i_1}\cdots c_{i_{k-1}}a_{i_{k+1}}\cdots a_{i_s} \right]$$

$$= \frac{1}{\prod_{j=1}^{s} a_{i_j}c_{i_j}} \left[a_{i_1}a_{i_3}a_{i_4}\cdots a_{i_s} + a_{i_4}a_{i_5}\cdots a_{i_s}c_{i_2}(a_{i_1} + c_{i_3}) + \sum_{k=4}^{s} c_{i_1}\cdots c_{i_{k-1}}a_{i_{k+1}}\cdots a_{i_s} \right]$$

$$= \frac{1}{\prod_{j=1}^{s} a_{i_j}c_{i_j}} \left[a_{i_1}\left(\sum_{k=2}^{s} c_{i_2}c_{i_3}\cdots c_{i_{k-1}}a_{i_{k+1}}\cdots a_{i_s} \right) + c_{i_2}c_{i_3}\cdots c_{i_s} \right]$$

$$= \frac{1}{c_{i_1}} \left[\sum_{k=2}^{s} \frac{1}{a_{i_2}a_{i_3}\cdots a_{i_k}c_{i_k}c_{i_{k+1}}\cdots c_{i_s}} \right] + \frac{1}{c_{i_1}a_{i_1}a_{i_2}\cdots a_{i_s}},$$

and using induction on the term in brackets

$$\sum_{k=1}^{s} \frac{1}{a_{i_1}a_{i_2}\cdots a_{i_k}c_{i_k}c_{i_{k+1}}\cdots c_{i_s}}$$

$$= \frac{1}{c_{i_1}} \left[\sum_{k=2}^{s} \frac{1}{c_{i_2}c_{i_3}\cdots c_{i_k}a_{i_k}a_{i_{k+1}}\cdots a_{i_s}} \right] + \frac{1}{c_{i_1}a_{i_1}\cdots a_{i_s}}$$

$$= \sum_{k=1}^{s} \frac{1}{c_{i_1}c_{i_2}\cdots c_{i_k}a_{i_k}a_{i_{k+1}}\cdots a_{i_s}}. \qquad \square$$

Now given any set $\{i_1, i_2, \ldots, i_s\} \subseteq \{\mathfrak{a}, \mathfrak{a}+1, \ldots, \alpha - 2\}$, where $i_1 = \mathfrak{a}$, consider all the terms of (5.7) having the form

$$\left(\frac{1}{A'_k C'_k}\right)_{\mathrm{L}} = \frac{1}{a_{\mathfrak{a}} a_{i_2} \cdots a_{i_k} c_{i_k} c_{i_{k+1}} \cdots c_{i_s} c_{\alpha-1}}, \qquad 1 \le k \le s.$$

Using Lemma 7 and $h^*_{\mathfrak{a}\alpha-2} - 1 = a_{\mathfrak{a}} + c_{\alpha-1}$ we can write

$$\begin{aligned}\sum_{k=1}^{s}\left(\frac{1}{A'_k C'_k}\right)_{\mathrm{L}} &= \frac{1}{h^*_{\mathfrak{a}\alpha-2} - 1}\cdot\sum_{k=1}^{s}\frac{a_{\mathfrak{a}} + c_{\alpha-1}}{a_{\mathfrak{a}} a_{i_2} \cdots a_{i_k} c_{i_k} c_{i_{k+1}} \cdots c_{i_s} c_{\alpha-1}}\\ &= \frac{1}{h^*_{\mathfrak{a}\alpha-2} - 1}\cdot\sum_{k=1}^{s}\left(\frac{1}{a_{i_2} \cdots a_{i_k} c_{i_k} \cdots c_{\alpha-1}} + \frac{1}{a_{\mathfrak{a}} a_{i_2} \cdots a_{i_k} c_{i_k} \cdots c_{i_s}}\right)\\ &= \frac{1}{h^*_{\mathfrak{a}\alpha-2} - 1}\cdot\sum_{k=1}^{s}\left(\frac{1}{a_{i_2} \cdots a_{i_k} c_{i_k} \cdots c_{\alpha-1}} + \frac{1}{c_{\mathfrak{a}} c_{i_2} \cdots c_{i_k} a_{i_k} \cdots a_{i_s}}\right),\end{aligned}$$

where, by induction,

$$\frac{1}{a_{i_2} \cdots a_{i_k} c_{i_k} \cdots c_{\alpha-1}}$$

appears in $\mathrm{prob}_{\mathrm{B}}(\alpha\beta \mid i_2\alpha - 2)$ and

$$\frac{1}{c_{\mathfrak{a}} c_{i_2} \cdots c_{i_k} a_{i_k} \cdots a_{i_s}}$$

appears in $\mathrm{prob}_{\mathrm{B}}(\alpha\beta \mid \mathfrak{a}\alpha - 1)$. Again both terms will be found in (5.13) and $(1/A'_k C'_k)_{\mathrm{L}}$ has a factor of $1/a_{\mathfrak{a}} c_{\alpha-1}$.

Finally the terms $(1/A'_i C'_i)_{\mathrm{T}}$ of (5.7) are handled in the same fashion as those of the form $(1/A'_i C'_i)_{\mathrm{L}}$. All the formulas merely contain an extra factor of $1/a_{\alpha-1}$. Now we have shown that all the terms of (5.7) occur in (5.13). It is also necessary to check that these are the only terms and that they each occur exactly once. This tedious but straightforward verification is left to the reader.

(4) We prove (5.9) by simultaneous reverse induction on $\mathfrak{a}$ and $\mathfrak{b}$. The anchor and induction steps are practically the same so only the latter is presented. By definition of the algorithm

$$\mathrm{prob}_{\mathrm{B}}(\alpha\beta \mid \mathfrak{a}\mathfrak{b}) = \frac{1}{h^*_{\mathfrak{a}\mathfrak{b}} - 1}\left[\sum_{\substack{(i,j)\in H_{\mathfrak{a}\mathfrak{b}}\\ (i,j)\neq(\mathfrak{a},\mathfrak{b})}} \mathrm{prob}_{\mathrm{B}}(\alpha\beta \mid ij)\right].$$

Using cases (1), (2), (3) and the induction hypotheses on $\mathfrak{a}$ and $\mathfrak{b}$ we can rewrite this as

$$\begin{aligned}\operatorname{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}\mathfrak{b}) &= \frac{1}{h^*_{\mathfrak{a}\mathfrak{b}}-1}\sum_i\left(\frac{1}{A_i'''C_i'''}\right)_{\mathrm{U}}\\ &= \frac{1}{h^*_{\mathfrak{a}\mathfrak{b}}-1}\left(\sum_i\frac{1}{A_i'''C_i'''}\right)_{\mathrm{U}},\end{aligned}\tag{5.14}$$

where the sum is over all $1/A_i'''C_i'''$ that appear in

$$\operatorname{prob}_{\mathrm{B}}(\alpha\beta|i'j') \qquad \text{for} \qquad (i',j')\in H^*_{\mathfrak{a}+1\,\mathfrak{b}+1},\ (i',j')\neq(\mathfrak{a}+1,\mathfrak{b}+1).$$

Hence

$$\begin{aligned}\sum_i\frac{1}{A_i'''C_i'''} &= (h^*_{\mathfrak{a}+1\,\mathfrak{b}+1}-1)\operatorname{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}+1\ \mathfrak{b}+1)\\ &= (h^*_{\mathfrak{a}+1\mathfrak{b}+1}-1)\sum_i\frac{1}{A_i''C_i''}.\end{aligned}\tag{5.15}$$

Furthermore the "in addition" clauses in (1), (2), (3), and inductively (4) allow us to index the terms $1/A_i''C_i''$ so that $1/a_{\mathfrak{a}+1}c_{\mathfrak{b}+2}$ is a factor of $1/A_i''C_i''$ for all $i\le I$ and $1/a_{\mathfrak{b}+2}c_{\mathfrak{a}+1}$ is a factor of $1/A_i''C_i''$ for all $i>I$. Since $h^*_{\mathfrak{a}+1\,\mathfrak{b}+1}-1=a_{\mathfrak{a}+1}+c_{\mathfrak{b}+2}=a_{\mathfrak{b}+2}+c_{\mathfrak{a}+1}$, (5.15) becomes

$$\begin{aligned}\sum_i\frac{1}{A_i'''C_i'''} &= (h^*_{\mathfrak{a}+1\,\mathfrak{b}+1}-1)\left[\sum_{i\le I}\frac{1}{A_i''C_i''}+\sum_{i>I}\frac{1}{A_i''C_i''}\right]\\ &= \sum_{i\le I}\frac{a_{\mathfrak{a}+1}}{A_i''C_i''}+\sum_{i\le I}\frac{c_{\mathfrak{b}+2}}{A_i''C_i''}+\sum_{i>I}\frac{a_{\mathfrak{b}+2}}{A_i''C_i''}+\sum_{i>I}\frac{c_{\mathfrak{a}+1}}{A_i''C_i''},\end{aligned}$$

where the numerator cancels into the denominator in each of the four sums. Hence we can substitute this expression into (5.14):

$$\begin{aligned}\operatorname{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}\mathfrak{b}) &= \frac{1}{h^*_{\mathfrak{a}\mathfrak{b}}-1}\left(\sum_i\frac{1}{A_i'''C_i'''}\right)_{\mathrm{U}}\\ &= \frac{1}{h^*_{\mathfrak{a}\mathfrak{b}}-1}\left[\sum_{i\le I}\frac{a_{\mathfrak{a}+1}}{A_i''C_i''}+\sum_{i\le I}\frac{c_{\mathfrak{b}+2}}{A_i''C_i''}+\sum_{i>I}\frac{a_{\mathfrak{b}+2}}{A_i''C_i''}+\sum_{i>I}\frac{c_{\mathfrak{a}+1}}{A_i''C_i''}\right]_{\mathrm{U}}\\ &= \frac{1}{h^*_{\mathfrak{a}\mathfrak{b}}-1}\left[\sum_{i\le I}\left(\frac{a_{\mathfrak{a}+1}}{A_i''C_i''}\right)_{\mathrm{U}}+\sum_{i\le I}\left(\frac{c_{\mathfrak{b}+2}}{A_i''C_i''}\right)_{\mathrm{U}}\right.\\ &\qquad\left.+\sum_{i>I}\left(\frac{a_{\mathfrak{b}+2}}{A_i''C_i''}\right)_{\mathrm{U}}+\sum_{i>I}\left(\frac{c_{\mathfrak{a}+1}}{A_i''C_i''}\right)_{\mathrm{U}}\right]\end{aligned}$$

$$= \frac{1}{h^*_{\mathfrak{ab}} - 1}\left[\sum_{i \le I} a_{\mathfrak{a}}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{N}} + a_{\mathfrak{a}}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{R}}\right.$$

$$+ c_{\mathfrak{a}}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{L}} + c_{\mathfrak{a}}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{T}}$$

$$+ \sum_{i \le I} c_{\mathfrak{b}+1}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{N}} + c_{\mathfrak{b}+1}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{R}}$$

$$+ a_{\mathfrak{b}+1}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{L}} + a_{\mathfrak{b}+1}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{T}}$$

$$+ \sum_{i > I} a_{\mathfrak{b}+1}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{N}} + a_{\mathfrak{b}+1}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{R}}$$

$$+ c_{\mathfrak{b}+1}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{L}} + c_{\mathfrak{b}+1}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{T}} + \sum_{i > I} c_{\mathfrak{a}}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{N}}$$

$$\left. + c_{\mathfrak{a}}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{R}} + a_{\mathfrak{a}}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{L}} + a_{\mathfrak{a}}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{T}}\right]$$

$$= \frac{1}{h^*_{\mathfrak{ab}} - 1}\left[(a_{\mathfrak{a}} + c_{\mathfrak{b}+1})\left\{\sum_{i \le I}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{N}} + \left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{R}}\right\}\right.$$

$$+ (a_{\mathfrak{b}+1} + c_{\mathfrak{a}})\left\{\sum_{i \le I}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{L}} + \left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{T}}\right\}$$

$$+ (\mathrm{a}_{\mathfrak{b}+1} + c_{\mathfrak{a}})\left\{\sum_{i > I}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{N}} + \left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{R}}\right\}$$

$$\left. + (a_{\mathfrak{a}} + c_{\mathfrak{b}+1})\left\{\sum_{i > I}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{L}} + \left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{T}}\right\}\right]$$

$$= \sum_{i \le I}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{N}} + \left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{R}} + \left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{L}} + \left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{T}}$$

$$+ \sum_{i > I}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{N}} + \left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{R}} + \left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{L}} + \left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{T}}$$

$$= \sum_{i \le I}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{U}} + \sum_{i > I}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{U}}$$

$$= \sum_{i}\left(\frac{1}{A_i'' C_i''}\right)_{\mathrm{U}} \qquad \text{as desired.}$$

Now that we have proved (1), (2), (3), and (4) the lemma is almost complete. In fact (5.1) follows immediately from (5.4), (5.6), (5.7), and (5.9). Reverse induction on $\mathfrak{a}$ and (5.1) will give us (5.2) for

$$\begin{aligned}
\sum_{\mathfrak{b}=\mathfrak{a}}^{\beta} \mathrm{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}\mathfrak{b}) &= \sum_i \left(\frac{1}{A_iC_i}\right)_{\mathrm{U}} = \left(\sum_i \frac{1}{A_iC_i}\right)_{\mathrm{U}} \\
&= \left[\sum_{\mathfrak{b}=\mathfrak{a}+1}^{\beta} \mathrm{prob}_{\mathrm{B}}(\alpha\beta|\mathfrak{a}+1\mathfrak{b})\right]_{\mathrm{U}} \\
&= \left[\left(\frac{1}{a_{\mathfrak{a}+1}} + \frac{1}{c_{\mathfrak{a}+1}} + \frac{1}{a_{\mathfrak{a}+1}c_{\mathfrak{a}+1}}\right) \cdot \prod_{\mathfrak{a}+1<i<\alpha} \left(1+\frac{1}{a_i}\right)\left(1+\frac{1}{c_i}\right)\right]_{\mathrm{U}} \\
&= \left(\frac{1}{a_{\mathfrak{a}}} + \frac{1}{c_{\mathfrak{a}}} + \frac{1}{a_{\mathfrak{a}}c_{\mathfrak{a}}}\right) \cdot \prod_{\mathfrak{a}<i<\alpha} \left(1+\frac{1}{a_i}\right)\left(1+\frac{1}{c_i}\right). \qquad \square
\end{aligned}$$

This is a very complicated proof, especially in view of the simplicity of the end result, Eq. (5.1). It would be interesting to find a more direct method for establishing Lemma 6.

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