THREE-MANIFOLDS OF CONSTANT VECTOR CURVATURE ONE

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ABSTRACT. A Riemannian manifold has $\mathrm{CVC}(\epsilon)$ if its sectional curvatures satisfy $\sec \leq \varepsilon$ or $\sec \geq \varepsilon$ pointwise, and if every tangent vector lies in a tangent plane of curvature ε. We present a construction of an infinite-dimensional family of compact $\mathrm{CVC}(1)$ three-manifolds.

Une variété riemannienne a $\mathrm{CVC}(\epsilon)$ lorsque $\sec \leq \varepsilon$ ou $\sec \geq \varepsilon$ pointwise, et chaque vecteur tangent se trouve dans une courbure ε plan. Nous construisons une famille infinie-dimensionnelle de compacts trois-variétés avec $\mathrm{CVC}(1)$.

1. **Introduction**

A Riemannian manifold has *constant vector curvature* ε if every tangent vector lies in a 2-plane of curvature ε and has *pointwise extremal curvature* ε if the sectional curvatures satisfy $\sec \geq \varepsilon$ or $\sec \leq \varepsilon$ pointwise. A manifold has $\mathrm{CVC}(\varepsilon)$ when it has both constant vector curvature ε and pointwise extremal curvature ε.

The study of $\mathrm{CVC}(\varepsilon)$ manifolds began with [12], motivated by rank-rigidity theorems as in [1, 2, 3, 4, 5, 6, 8, 11, 15, 16, 17]. Classification results in [12] demonstrate the rigid nature of *finite volume* $\mathrm{CVC}(\varepsilon)$ three-manifolds with $\varepsilon \leq 0$. When $\varepsilon = -1$, they are all locally homogeneous. When $\varepsilon = 0$, components of non-isotropic points admit Riemannian product decompositions. These rigidity results fail without the finite volume assumption by [7, 13, 14].

Here, we illustrate the relative flexibility of this curvature condition when $\varepsilon > 0$. We construct an infinite-dimensional family of *compact* $\mathrm{CVC}(1)$ three-manifolds. These manifolds also satisfy the following spherical rank condition: Each geodesic $\gamma(t)$ admits a Jacobi field $J(t)$ with $\sec(\dot{\gamma}, J)(t) \equiv 1$. Contrastingly, in dimension three, only the spherical space forms satisfy the (a posteriori more stringent) spherical rank condition obtained by replacing Jacobi fields with parallel fields [6].

Our construction ”deforms” compact locally homogeneous three-manifolds admitting a Riemannian submersion to a constant curvature surface. For $c \in \mathbb{R}$, let G denote $\mathrm{SU}(2)$, the Heisenberg group, or $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ when $c < 1$, $c = 1$, or $c > 1$, respectively. Let Γ be a cocompact lattice in G. The parameter c and lattice Γ determine the deformed Riemannian submersion:

The group G admits a left-invariant framing $\{e_1, e_2, e_3\}$ with

$$[e_1, e_2] = -2e_3, \qquad [e_1, e_3] = (1-c)e_2, \qquad [e_2, e_3] = -(1-c)e_1.$$

This framing is orthonormal for a metric satisfying

(1) every tangent plane containing the vector e_3 has curvature 1,
(2) the tangent plane spanned by e_1 and e_2 has curvature $\lambda = -(2c+1)$,
(3) all sectional curvatures lie between 1 and λ, and

The first author was partially supported by NSF grant DMS-1207655.

(4) the vector field e_3 is Killing.

By (1) and (3), the metric Lie group G has $\mathrm{CVC}(1)$. By (4), the e_3-orbit space Σ admits a metric making the orbit map $G \to \Sigma$ a Riemannian submersion; this metric has constant Gaussian curvature $K = \lambda + 3 = 2(1-c)$ by [10].

The lattice Γ acts by isometric left-translations on G with compact locally homogeneous quotient (M_c, g_0). The invariant framing $\{e_1, e_2, e_3\}$ induces an orthonormal framing $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ of (M_c, g_0) satisfying (1)-(4) above. Up to a finite cover of M_c, $\bar{e}_3$ generates a free circle action, inducing a Riemannian submersion $\pi : (M_c, g_0) \to (S_c, s_0)$ with target a compact surface of constant curvature $2(1-c)$.

We regard (M_c, g_0) as a "model" $\mathrm{CVC}(1)$ three-manifold. The CVC *Transform* presented below deforms g_0 into a family of locally inhomogeneous $\mathrm{CVC}(1)$ metrics on M_c parameterized by a function space on S_c. While this construction shows that locally inhomogeneous $\mathrm{CVC}(1)$ metrics abound, preliminary analysis suggests that the following uniformization conjecture holds:

Conjecture: *If (M, g) is a closed $\mathrm{CVC}(1)$ three-manifold, then the underlying smooth manifold M is a locally homogeneous space and admits a locally homogenous $\mathrm{CVC}(1)$ metric as described above.*

2. Frame certification of $\mathrm{CVC}(1)$

Let $\{w_i\}_{i=1}^3$ be an orthonormal framing of (X^3, g) satisfying:

$$[w_1, w_2] = \alpha w_1 + \beta w_2 - 2w_3, \quad [w_1, w_3] = k w_2, \quad [w_2, w_3] = -k w_1, \tag{2.1}$$

with α, β smooth functions on X and $k \in \mathbb{R}$. By Koszul's formula,

$$\begin{aligned} \nabla_{w_1} w_3 &= w_2 & \nabla_{w_2} w_3 &= -w_1 \\ \nabla_{w_3} w_1 &= (1-k) w_2 & \nabla_{w_3} w_2 &= -(1-k) w_1 \\ \nabla_{w_2} w_1 &= -\beta w_2 + w_3 & \nabla_{w_2} w_2 &= \beta w_1 \\ \nabla_{w_1} w_2 &= \alpha w_1 - w_3 & \nabla_{w_1} w_1 &= -\alpha w_2 \end{aligned} \tag{2.2}$$
$$\nabla_{w_3} w_3 = 0.$$

By (2.2), the Laplacian $\Delta = \Sigma_i\, w_i w_i - \nabla_{w_i} w_i$ and curvature components $R_{ijkl} = g(\nabla_{w_i} \nabla_{w_j} w_k - \nabla_{w_j} \nabla_{w_i} w_k - \nabla_{[w_i, w_j]} w_k, w_l)$ simplifiy as

$$\Delta = w_1 w_1 + w_2 w_2 + w_3 w_3 - \beta w_1 + \alpha w_2, \tag{2.3}$$

$$R_{1221} = (2k-3) - (w_2(\alpha) - w_1(\beta) + \alpha^2 + \beta^2), \tag{2.4}$$

$$R_{1331} = R_{2332} = 1, \tag{2.5}$$

$$R_{1213} = R_{1223} = R_{1323} = 0. \tag{2.6}$$

The symmetries $R_{ijkl} = R_{klij} = -R_{jikl}$ determine the remaining components.

Lemma 2.1. *A 2-plane with unit-normal vector $n = \Sigma_{i=1}^3 c_i w_i$ has sectional curvature* $\sec = c_1^2 + c_2^2 + c_3^2\, R_{1221}$.

Proof. By (2.6), $\{w_i\}$ diagonalizes Ricci. Now substitute (2.5) into [12, Lemma 2.2]. $\square$

Proposition 2.2. *If* (X^3, g) *admits an orthonormal framing as in (2.1), then*
(1) (X, g) *is* $\mathrm{CVC}(1)$,
(2) w_3 *is Killing.*
(3) *Each geodesic* $\gamma(t)$ *in* X *admits a Jacobi field* $J(t)$ *with* $\sec(\dot{\gamma}, J)(t) \equiv 1$.

Proof. By Lemma 2.1, the sectional curvatures lie between 1 and R_{1221} pointwise, and every tangent 2-plane containing the vector w_3 has curvature one. Proposition-(1) follows. By (2.2), $v \mapsto \nabla_v w_3$ is skew-symmetric, implying Propositon-(2). As Killing fields restrict to Jacobi fields, Proposition-(3) is immediate for geodesics that are *not* tangent to w_3.

For a geodesic $\gamma(t)$ tangent to w_3, first use the fact that if $\{x, y, w_3\}$ is an orthonormal frame at a point, then the function

$$R(\cos(t)x + \sin(t)y, w_3, w_3, \cos(t)x + \sin(t)y)$$

is identically one from which it follows that $R(x, w_3)w_3 = x$. Now if $V(t)$ is a unit-orthogonal and parallel field along $\gamma(t)$, then $J(t) = (\cos(t) + \sin(t))V(t)$ is a Jacobi field with the desired property. □

3. The CVC Transform

Let $\pi : (M_c, g_0) \to (S_c, s_0)$ and $\{\bar{e}_i\}_{i=1}^3$ be as in the introduction. Then

$$[\bar{e}_1, \bar{e}_2] = -2\bar{e}_3, \qquad [\bar{e}_1, \bar{e}_3] = (1-c)\bar{e}_2, \qquad [\bar{e}_2, \bar{e}_3] = -(1-c)\bar{e}_1. \tag{3.1}$$

This framing satisfies (2.1) with $\alpha = \beta = 0$ and $k = (1-c)$. For $h \in C^\infty(S_c)$, let $s_h = e^{-2h}s_0$. The Gaussian curvature of s_h is

$$K_h = e^{2h}(\Delta_{s_0} h + 2(1-c)),$$

where Δ_{s_0} is the Laplacian for (S_c, s_0). By (2.3), the Laplacian of (M_c, g_0) is given by

$$\Delta_{g_0} = \bar{e}_1\bar{e}_1 + \bar{e}_2\bar{e}_2 + \bar{e}_3\bar{e}_3. \tag{3.2}$$

For each $\phi \in C^\infty(S_c)$,

$$\Delta_{g_0} \pi^*(\phi) = \pi^*(\Delta_{s_0}\phi). \tag{3.3}$$

Let ds_0 denote the Riemannian area form for s_0 and define

$$\mathcal{F} = \{h \in C^\infty(S_c) \mid \int_{S_c} (1 - e^{-2h})\, ds_0 = 0\}.$$

For $h \in \mathcal{F}$ there exists $f \in C^\infty(S_c)$ such that

$$\Delta_{s_0} f = 2(1 - e^{-2h}). \tag{3.4}$$

The derivation e_3 annihilates $H = \pi^*(h)$, $F = \pi^*(f)$, and $G = H + (1-c)F$.

Definition 3.1. The CVC-transform of g_0 determined by $h \in \mathcal{F}$ is the orthonormalizing metric for the framing

$$e_1 = e^H(\bar{e}_1 - \bar{e}_2(F)\bar{e}_3), \;\; e_2 = e^H(\bar{e}_2 + \bar{e}_1(F)\bar{e}_3), \;\; e_3 = \bar{e}_3. \tag{3.5}$$

Given $h \in \mathcal{F}$, let g_h denote the CVC-transform of g_0 determined by h.

Proposition 3.1. *Let $\pi : (M_c, g_0) \to (S_c, s_0)$ be a locally homogeneous Riemannian submersion as described above. For each $h \in \mathcal{F}$, the* CVC*-transform g_h of g_0 satisfies*

(1) *The map π is a Riemannian submersion between (M_c, g_h) and (S_c, s_h),*
(2) *The three-manifold (M_c, g_h) has* $\mathrm{CVC}(1)$ *with scalar curvature function $S_h = 2\lambda_h + 4$ where $\lambda_h = \pi^*(K_h) - 3$.*
(3) *Each complete geodesic $\gamma(t)$ in (M_c, g_h) admits a Jacobi field $J(t)$ with $\sec(\dot\gamma, J)(t) \equiv 1$.*

Proof. Let $\{e_i\}_{i=1}^3$ be the orthonormal framing for g_h defined in (3.5). Part (1) of the Proposition is immediate from the fact that $e_3 = \bar e_3$ and (3.5).

As a preliminary step in proving part (2) of the Proposition, use (3.2-3.4) to deduce

$$\bar e_1(\bar e_1(F)) + \bar e_2(\bar e_2(F)) = 2(1 - e^{-2H}). \tag{3.6}$$

Routine, but tedious, calculations using (3.1), (3.5), and (3.6) imply

$$[e_1, e_2] = -e_2(G)e_1 + e_1(G)e_2 - 2e_3, \quad [e_1, e_3] = (1-c)e_2, \quad [e_2, e_3] = -(1-c)e_1.$$

These bracket relations and Proposition 2.2-(1) show that (M_c, g_h) has $\mathrm{CVC}(1)$. To evaluate its scalar curvature, first set $\lambda_h = \sec(e_1, e_2)$. By (2.4-2.5), it suffices to prove that $\lambda_h = \pi^*(K_h) - 3$, where $K_h = e^{2h}(\Delta_{\bar s} h + 2(1-c))$ is the Gaussian curvature of (S_c, s_h). By [10], $\pi^*(K_h) = \lambda_h + \frac{3}{4}\|[e_1, e_2]^v\|^2 = \lambda_h + 3$, concluding the proof of part (2) of the Proposition.

Part (3) of the Proposition is immediate from Proposition 2.2-(3), concluding the proof.

□

Remark 3.1. The function space $\mathcal{F}$ corresponds with the quotient of $C^\infty(S_c)$ by the constant functions. For $f \in C^\infty(S_c)$, let $A_f = \mathrm{Area}(S_c, s_f)$. The map $C^\infty(S_c) \to \mathcal{F}$ defined by $g \mapsto g - \frac{\ln(A_0) - \ln(A_f)}{2}$ is the natural bijection.

Remark 3.2. If $h_0, h_1 \in \mathcal{F}$ and $s \in [0,1]$, then $h_s = -\frac{1}{2}\ln((1+s)e^{-2h_0} + se^{-2h_1}) \in \mathcal{F}$. It follows that the space of transformed metrics $\{g_h \mid h \in \mathcal{F}\}$ is path-connected.

Remark 3.3. The authors of [9] prescribe K_h in the conformal class of s_0, up to a diffeomorphism of S_c and the Gauss-Bonnet obstruction. As such, there is considerable freedom in prescribing the scalar curvatures of compact $\mathrm{CVC}(1)$ three-manifolds.

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