

TWO REMARKS ON THE LENGTH SPECTRUM OF A RIEMANNIAN MANIFOLD

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ABSTRACT. We demonstrate that every closed manifold of dimension at least two admits smooth metrics with respect to which the length spectrum is not a discrete subset of the real line. In contrast, we show that the length spectrum of any real analytic metric on a closed manifold is a discrete subset of the real line. In particular, the length spectrum of any closed locally homogeneous space forms a discrete set.

1. INTRODUCTION

It is widely believed that a closed Riemannian manifold of dimension at least two has infinitely many prime closed geodesics. The results of numerous authors verify this belief for large classes of manifolds as summarized in [Ta, Section 5].

While considering this problem, Abraham introduced the notion of a *bumpy metric* [A]. The main result of [A] is that the collection of smooth bumpy metrics on a closed manifold forms a residual set in the space of all smooth metrics on M (see [An] for a complete proof); he conjectured that such metrics should have infinitely many prime closed geodesics. Abraham also showed that if g is a bumpy metric on a closed manifold M , then for any $L > 0$ the collection of its geometrically distinct closed geodesics of length less than L is finite. Hence, we see that generically, the *length spectrum*—the collection of lengths of all smoothly closed geodesics—is a discrete subspace of the real line.

To our knowledge, examples of closed manifolds with indiscrete length spectrum do not appear in the literature, motivating the following:

1.1. Proposition. *Let M be a smooth manifold. If M is of dimension at least three, then for any smooth simple closed curve $\gamma : S^1 \rightarrow M$ there is a smooth metric g on M with respect to which γ is a closed geodesic and its length $l_g(\gamma)$ is an accumulation point of the length spectrum. In dimension two the same result holds if the tubular neighborhood of γ is an oriented disk bundle.*

Given Abraham’s bumpy metric theorem and the previous proposition one is led to consider whether there are additional large classes of metrics for which the length spectrum is discrete.

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For a real analytic manifold M , the dense subset of real analytic metrics in the space of smooth metrics form such a class:

1.2. Proposition. *Let M be a closed real analytic manifold and g a real analytic metric on M . Then the length spectrum of (M, g) forms a discrete subset of the real line.*

2. THE PROOFS

Proof of Proposition 1.1. Let M be a closed manifold of dimension $n \geq 3$, let $S^1 = \mathbb{R}/\mathbb{Z}$ denote the standard circle and let $\gamma : S^1 \rightarrow M$ be a smooth simple closed curve: such curves exist in every free homotopy class. The tubular neighborhood theorem [Hi, Thm. 5.2, Ch. 4] says that there is a smooth vector bundle N over S^1 of rank $(n - 1)$ along with a diffeomorphism $i : N \rightarrow \mathcal{T} \subset M$ onto an open subset \mathcal{T} of M , so that its composition with the zero section $S^1 \rightarrow N$ is the map γ .

We now recall that up to isomorphism there are exactly two vector bundles of rank $n - 1$ over S^1 : one orientable and one non-orientable [Hi, Ch. 4, Sec. 4, Ex. 2], [Ra, Chp. 5]. To facilitate constructing the metric g , we let B^{n-1} be the standard open unit ball around 0 in \mathbb{R}^{n-1} , which is diffeomorphic to each fiber of the vector bundle N over S^1 . Now consider the quotient $(B^{n-1} \times \mathbb{R})/\mathbb{Z}$, where $z \in \mathbb{Z}$ acts by sending $((x_1, x_2, \dots, x_{n-1}), t)$ to $((x_1, x_2, \dots, x_{n-1}), t + z)$ if N is orientable, and to $((-1)^z x_1, x_2, \dots, x_{n-1}), t + z)$ if N is non-orientable. We then have the following commutative diagram:

$$\begin{array}{ccccc}
 & & S^1 & & \\
 & \text{0-section} & \downarrow & \gamma & \\
 t \mapsto (0, t) & \curvearrowright & & & \\
 (B^{n-1} \times \mathbb{R})/\mathbb{Z} & \xrightarrow{\sim} & N & \xrightarrow[\sim]{i} & \mathcal{T}
 \end{array}$$

where the horizontal maps are diffeomorphisms.

The standard metric $h_0 = h_0^B \oplus h_0^{\mathbb{R}}$ on $B^{n-1} \times \mathbb{R}$ induces a Riemannian metric on \mathcal{T} that we also denote by h_0 . Now, let $\{a_k\}_{k \geq 2}$ be a sequence of numbers greater than 1 so that $a_k \rightarrow 1$, and for each $k \geq 2$ we let $c_k = (0, \frac{1}{k}, 0, \dots, 0) \in B^{n-1}$, $\epsilon_k = \frac{1}{2k(k+1)}$ and let $B_k = B(c_k; \epsilon_k)$ be the ball of radius ϵ_k centered at c_k . Then for each $k \geq 2$ we let $f_k : B_k \rightarrow [1, a_k]$ be a smooth (radially symmetric) bump function with $f_k \equiv a_k$ on $B(c_k; \frac{1}{4}\epsilon_k)$ and $f_k \equiv 1$ on $B(c_k; \epsilon_k) - B(c_k; \frac{3}{4}\epsilon_k)$. Now, choosing the sequence $\{a_k\}$ so that a_k converges to 1 sufficiently fast, we obtain a smooth bounded function $f : B^{n-1} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x \in B^{n-1} - \cup_{k \in \mathbb{N}} B_k \\ f_k(x) & x \in B_k \end{cases}$$

We may then define a \mathbb{Z} -invariant metric h on $B^{n-1} \times \mathbb{R}$ via

$$h(x; t) = (h_0^B)_x \oplus f(x)(h_0^{\mathbb{R}})_t.$$

Then h induces a smooth metric—which we also denote by h —on the tubular neighborhood \mathcal{T} .

For each $k \geq 2$, let $\gamma_k : S^1 \rightarrow \mathcal{T}$ be the smoothly closed loop induced by the smooth curve $\tilde{\gamma}_k : \mathbb{R} \rightarrow B^{n-1} \times \mathbb{R}$ given by $t \mapsto (c_k; t)$. Then with respect to the metric h we see that $\gamma_k : S^1 \rightarrow (\mathcal{T}, h)$ is a geodesic with $l_h(\gamma_k) = a_k$. Now, let $r : B^{n-1} \rightarrow [0, \infty)$ be the smooth map given by $r(x) = \|x\|^2$ and for any $\epsilon > 0$, let $\mathcal{T}_\epsilon \equiv \{(x; t) \in \mathcal{T} : r(x) < \epsilon\}$. We then obtain a smooth metric g on M given by

$$g_p = \begin{cases} (\theta(r(x))h + (1 - \theta(r(x)))g_0 & p = \overline{(x; t)} \in \mathcal{T} \\ g_0 & p \in M - \mathcal{T} \end{cases}$$

where g_0 is some background metric on M and $\theta : [0, 1] \rightarrow [0, 1]$ is smooth with $\theta(t) = 1$ for $0 \leq t \leq 2/5$ and $\theta(t) = 0$ for $3/5 \leq t \leq 1$. Then on $\mathcal{T}_{2/5}$ we see that $g = h$. Hence, $\gamma : S^1 \rightarrow (M, g)$ is a geodesic with $l_g(\gamma) = l_h(\gamma) = 1$ and, for all $k \geq 2$, $\gamma_k : S^1 \rightarrow (M, g)$ is a closed geodesic (freely homotopic to γ) with $l_g(\gamma_k) = l_h(\gamma_k) = a_k$. Hence, $l_g(\gamma)$ is an accumulation point of the length spectrum of (M, g) .

In dimension two, if one assumes the tubular neighborhood \mathcal{T} of γ is oriented (such curves exist), the above construction carries through without modification. \square

Example. Not only may the length spectrum fail to be discrete, it may even be uncountable as seen in this example. Let $K \subset [0, 1]$ denote the standard middle third Cantor set and let $f : [0, 1] \rightarrow \mathbb{R}$ be a non-negative smooth function with zero set K . Then the function $g : [0, 1] \rightarrow \mathbb{R}$ defined by $g(x) - 1 = \int_0^x f(t)dt$ has critical set K and is strictly increasing on K . Consider the metric of revolution on $[0, 1] \times S^1$ obtained by revolving the graph of g around the x -axis. For each $k \in K$ the set $\{k\} \times S^1$ is a closed geodesic of length $2\pi g(k)$. As K is uncountable and g increases on K , this metric has uncountable length spectrum.

Proof of Proposition 1.2. We begin by fixing some notation.

Let Ω be the set consisting of piecewise smooth loops in M . That is, $\omega : [0, 1] \rightarrow M$ is an element of Ω if and only if

- (1) $\omega(0) = \omega(1)$;
- (2) there is a subdivision $t_0 = 0 < t_1 < \dots < t_k = 1$ such that $\omega \upharpoonright [t_{i-1}, t_i]$ is smooth for each $i = 0, \dots, k-1$.

Given $\omega \in \Omega$ the tangent space to Ω at ω , denoted by $T_\omega\Omega$, is the collection of all vector fields W_i along ω which are smooth on the corresponding intervals $[t_{i-1}, t_i]$ for $i = 1, \dots, k$ and satisfy $W_0 = W_1$. By a *variation* of ω (through closed loops) we shall mean a map $\bar{\alpha} : (-\epsilon, \epsilon) \rightarrow \Omega$ such that

- (1) $\bar{\alpha}(0) = \omega$;
- (2) there is a partition $t_0 = 0 < t_1 < \dots < t_l = 1$ of $[0, 1]$ such that the map $\alpha : (-\epsilon, \epsilon) \times [0, 1] \rightarrow \Omega$, given by $\alpha(u, t) = \bar{\alpha}(u)(t)$, restricted to $(-\epsilon, \epsilon) \times [t_{i-1}, t_i]$ is smooth for each $i = 1, \dots, l$.

Given any variation $\bar{\alpha}$ of ω the vector field $W_t = \frac{\partial}{\partial u}\alpha(0, t)$ is in the tangent space $T_\omega\Omega$, and conversely any vector field $W_t \in T_\omega\Omega$ gives rise to a variation of the loop ω : $\alpha(u, t) = \exp_{\omega(t)}(uW_t)$.

We let $E : \Omega \rightarrow \mathbb{R}$ be the energy functional

$$E(\omega) \equiv \int_0^1 \|\omega'(t)\|^2 dt,$$

and let $L : \Omega \rightarrow \mathbb{R}$ be the length functional

$$L(\omega) = \int_0^1 \|\omega'(t)\| dt.$$

The Cauchy-Schwarz inequality implies that $L(\omega)^2 \leq E(\omega)$ with equality if and only if $\|\omega'(t)\|$ is constant. In particular, equality is obtained when ω is a geodesic. The first variation formula implies that $\omega \in \Omega$ is a closed geodesic in (M, g) if and only if it is a critical point of E ; that is, $\frac{d}{du}(E \circ \bar{\alpha})(0) = 0$ for any variation $\bar{\alpha}$.

Now for each $c > 0$ we let $\Omega^c = E^{-1}([0, c])$ and $\text{Int } \Omega^c = E^{-1}([0, c))$. The space Ω^c is infinite dimensional; however, following Milnor [M] we can construct a finite dimensional approximation to Ω^c as follows.

For each partition $t_0 = 0 < t_1 < \dots < t_k = 1$ of the unit interval $[0, 1]$ we let $\Omega(t_0, t_1, \dots, t_k)$ be the collection of $\omega : [0, 1] \rightarrow M$ such that

- (1) $\omega(0) = \omega(1)$
- (2) $\omega \upharpoonright [t_{i-1}, t_i]$ is a geodesic for $i = 0, 1, \dots, k-1$.

That is, $\Omega(t_0, t_1, \dots, t_k)$ is the space of piecewise geodesic loops. We let $\Omega(t_0, t_1, \dots, t_k)^c = \Omega^c \cap \Omega(t_0, t_1, \dots, t_k)$ and $\text{Int } \Omega(t_0, t_1, \dots, t_k)^c = (\text{Int } \Omega^c) \cap \Omega(t_0, t_1, \dots, t_k)$. Then as in Milnor we have the following lemma.

2.1. Lemma (cf. [M] Lem. 16.1 & Thm. 16.2). *Let (M, g) be a compact real analytic Riemannian manifold and let $c > 0$. Then for all finite partitions $t_0 = 0 < t_1 < \dots < t_k = 1$ with sufficiently small mesh the set $B \equiv \text{Int } \Omega(t_0, t_1, \dots, t_k)^c$ can be given the real analytic structure of an open subset of M^k in a natural way. Furthermore, the restriction of the energy functional E to B , which we will denote by \tilde{E} , has the following properties:*

- (1) \tilde{E} is analytic and given by $\tilde{E}(\omega) = \sum_{i=1}^k \frac{d_g(\omega(t_{i-1}), \omega(t_i))^2}{(t_i - t_{i-1})}$;
- (2) for each $0 < a < c$ the set $B^a \equiv \tilde{E}^{-1}([0, a])$ is a compact subset of B ;
- (3) the critical points of \tilde{E} on B coincide with those of E on $\text{Int } \Omega^c$.

Proof. Since M is compact we may fix an $\epsilon > 0$ so that whenever $x, y \in M$ are such that $d_g(x, y) < \epsilon$, then there is a unique minimizing geodesic from x to y of length less than ϵ and, since g is analytic, this geodesic will depend analytically on x and y (cf. [M, Lem. 10.3]).

Now take a subdivision $t_0 = 0 < t_1 < \dots < t_k = 1$ of the interval $[0, 1]$ with mesh less than ϵ^2/c . Then for any $\omega \in \text{Int } \Omega(t_0, t_1, \dots, t_k)^c$ we have

$$(2.2) \quad \begin{aligned} d_g(\omega(t_{i-1}), \omega(t_i))^2 &= (L_{t_{i-1}}^{t_i} \omega)^2 &= (t_i - t_{i-1}) E_{t_{i-1}}^{t_i} \omega \\ &\leq (t_i - t_{i-1}) E \omega \\ &< (t_i - t_{i-1}) c \\ &< \epsilon^2. \end{aligned}$$

Therefore by our choice of ϵ , the geodesic $\omega \upharpoonright [t_{i-1}, t_i]$ is uniquely and analytically determined by its endpoints, and we conclude that the map

$$\omega \in B \xrightarrow{\varphi} (\omega(t_0), \omega(t_1), \dots, \omega(t_{k-1})) \in M^k$$

defines a homeomorphism of B onto an open subset of M^k , which induces a real analytic structure on B .

The expression for \tilde{E} given in (1) follows from Equation 2.2 and the analyticity follows since the geodesics depend analytically on the endpoints. Property (2) follows from the fact that φ maps B^a homeomorphically onto the closed subset of M^k given by

$$\{(p_0, p_1, \dots, p_{k-1}) \in M^k : \sum_{i=1}^{k-1} \frac{d_g(p_{i-1}, p_i)^2}{(t_i - t_{i-1})} + \frac{d_g(p_{k-1}, p_0)^2}{(t_k - t_{k-1})} \leq a\},$$

which is compact since M is compact. Finally, statement three follows from the fact that each smoothly closed geodesic is a broken geodesic and the first variation formula, which implies that the critical points of \tilde{E} are the smooth geodesics. \square

Now, assuming the assumptions of the preceding lemma are obtained, for each $0 < a < c$, B^a is a compact subset of B that contains all the critical points of E on $\text{Int } \Omega^a = E^{-1}([0, a])$. We now recall the following theorem concerning the critical values of real analytic functions.

2.3. Theorem ([SS]). *Let $f : D \subset \mathbb{R}^N \rightarrow \mathbb{R}$ be a real analytic function on an open subset of euclidean N -space, and let Z be the collection of critical points of f on D . Then for any compact subset $K \subset D$ we have $f(Z \cap K)$ is finite. That is, the set of critical values of f restricted to a compact set is always finite.*

It follows that since $\tilde{E} : B \rightarrow \mathbb{R}$ has the same critical values as E and $B^a \subset B$ is compact, the collection of critical values of E on $\text{Int } \Omega^a = E^{-1}([0, a])$ is finite. That is, there are finitely many critical values of E less than a . Therefore, the length spectrum of (M, g) is a discrete subset of \mathbb{R} . \square

Remark. As a consequence of Proposition 1.2, a locally homogeneous Riemannian manifold has a discrete length spectrum, our original motivation for this note. A nearly identical proof gives the following analogue of Proposition 1.2 for non-compact real analytic manifolds: If M is a real analytic manifold, then an infinite set of closed geodesics with accumulating lengths cannot accumulate in M .

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