

WEAKLY HYPERBOLIC GROUP ACTIONS

by

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In loving memory of my grandfather, Milton Gerstine.

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CHAPTER I

Introduction

1.1 Dynamics in Broad Terms

Broadly speaking, this thesis contributes to the theory of dynamical systems. Classically, the subject of dynamics studies how the properties of some system, typically called phase space, evolves in time. Time can be continuous or discrete. By allowing negative time, one also takes into consideration what the system was like in the past.

Put in more mathematical terms, classical dynamics studies how a phase space evolves under a group action of the real numbers or of the integers. More recently, the study of dynamics has considered systems evolving in "generalized time". More precisely, modern dynamical systems may be regarded as the general study of infinite group actions on space. There is a wide array of different phenomena exhibited by dynamical systems, depending simultaneously on the phase space, the structures preserved by the system, and the generalized time or acting group.

For completeness, we begin with the definition of a group action and proceed to indicate how the space acted upon, the structures preserved by the action, and the acting group each contribute to the behavior of the dynamical system.

Definition I.1. Let G be a group and X a set. A *group action* of G on X is a group

homomorphism $\phi : G \rightarrow \text{Bij}(X)$, where $\text{Bij}(X)$ denotes the group of bijections of X .

Typically, one wants to restrict attention to group actions for which the kernel of the defining homomorphism is trivial. Such actions are said to be *effective* and for these actions, different group elements from G act by different bijections on X .

The type of dynamics one obtains may depend very much on the phase space X . If the phase space is large like the Euclidean plane, infinite orbits in the system may very well escape off to infinity. However, if the space is small like the sphere, there is no infinity for orbits to escape to, and consequently all infinite orbits of the system will be forced to accumulate. In this thesis we will consider dynamical systems defined on small phase spaces. More precisely, we will investigate weakly hyperbolic group actions which, by definition, will have *closed* (compact and without boundary) Riemannian manifolds as the phase space. For the weakly hyperbolic group actions of the integers, more commonly called the Anosov diffeomorphisms, it is conjecturally open that these dynamical systems can only have phase space finitely covered by a nilmanifold (see I.3, III.2, and [Mar99, Conjecture 12]).

The dynamical behavior exhibited by a system and the conclusions one may draw depend considerably on the structures preserved by the group action. For example, ergodic theory is the study of group actions $\phi : G \rightarrow \text{Aut}(X, \mu)$ on a measure space (X, μ) by measure preserving bijections. In this setting, the phase space being small means that the μ is a finite measure. Already in this case, one can draw non-trivial conclusions about how orbits recur in the system. Specifically, there is the well known Poincare recurrence theorem:

Theorem I.2 (Poincare). *Let T be a measure preserving transformation of a probability space (X, μ) and let $A \subset X$ be a measurable set with complement A^C . Then*

for any $N \in \mathbb{N}$

$$\mu(\{x \in A \mid \{T^n(x)\}_{n \geq N} \subset A^C\}) = 0.$$

In the present work, we will be considering weakly hyperbolic actions (III.1) that preserve a smooth (hence finite) measure on a closed Riemannian manifold. Though these actions will preserve additional structures, we will often use the existing results in ergodic theory to make preliminary conclusions in the measurable setting.

Smooth dynamical systems is the study of group actions $\phi : G \rightarrow \text{Diff}(M)$ on a smooth manifold M by diffeomorphisms. Attached to a smooth dynamical system is the induced derivative action $d(\phi) : G \rightarrow \text{Homeo}(TM)$ on the tangent bundle TM . In addition to preserving a smooth structure, the action may preserve a variety of other structures. For example, actions preserving affine connections, (pseudo)-Riemannian metrics, symplectic forms, and projective structures are all currently active areas of research.

The actions under consideration in this thesis will be required to have finitely many group elements that preserve continuous subbundles of the tangent bundle. One necessarily obtains topological restrictions on the phase spaces supporting such actions. In addition, it will be required that vectors in these invariant subbundles have lengths that contract exponentially in forward time. Such behavior belongs to the theory of hyperbolic dynamics. A very well studied class of hyperbolic dynamical systems are the Anosov diffeomorphisms. These diffeomorphisms leave invariant complementary subbundles, one of which consists of vectors that exponentially contract in length and the other of which consists of vectors that exponentially expand in length under iterations of the diffeomorphism.

Definition I.3. Let M be a closed Riemannian manifold. A diffeomorphism $f \in$

$\text{Diff}^k(M)$ ($k \geq 1$) is *Anosov* if there exist a continuous df -invariant splitting

$$TM = E_f^s \oplus E_f^u,$$

and a continuous Riemannian metric such that for each unit vector $v \in TM$:

- $\|df(v)\| < 1$ for $v \in E_f^s$
- $\|df(v)\| > 1$ for $v \in E_f^u$

Example I.4. Let $g \in SL(n, \mathbb{Z})$ be a matrix with no eigenvalues of unit modulus. Then g defines a smooth volume preserving Anosov diffeomorphism on \mathbb{T}^n . The required invariant distributions are given simply in terms of the eigenspaces of g .

As it turns out, all examples of Anosov diffeomorphisms on tori arise as in the last example. More specifically, the Franks/Manning continuous classification of Anosov diffeomorphism (IV.1 and IV.2) shows that any volume preserving Anosov diffeomorphism is up to a global continuous change of coordinates defined by a hyperbolic matrix in $SL(n, \mathbb{Z})$.

Weakly hyperbolic actions are one possible generalization of the property of a diffeomorphism being Anosov. Weak hyperbolicity relaxes the condition that a single diffeomorphism exponentially contracts and expands distances in all directions. Instead, weak hyperbolicity specifies finitely many diffeomorphisms in a subgroup of the group of diffeomorphisms that individually exponentially contract or expand some direction in phase space. Weak hyperbolicity also requires that all possible directions are in the span of these special contracting and expanding directions.

In addition to the topology of the phase space and the structures preserved by a dynamical system, the algebraic properties of the acting group also play a role in determining the possible group actions. The groups with little algebraic structure

are the free groups. Since these groups have no nontrivial relations between group elements, it is quite easy to construct group actions for these groups. Specifically, one only needs to associate to each generator of the free group a different bijection of the phase space preserving the required structures.

When the acting group has complicated relations between group elements, such as $SL(n, \mathbb{Z})$, one can imagine that it may be quite non-trivial, if not impossible, to realize these relations by bijections of phase space preserving a nontrivial structure. For example, it is conjectured that there are no effective actions of higher rank lattices (II.23) by homeomorphisms on the circle. One such result in this direction is the following:

Theorem I.5 ([WM94a], Witte-Morris). *Suppose that Γ is an arithmetic subgroup of a simple algebraic group over \mathbb{Q} with \mathbb{Q} -rank at least two. Then every continuous action of Γ on the real line or on the circle factors through an action of a finite quotient of Γ*

Intuitively, such large groups cannot act nontrivially on small spaces. Theorem I.5 is part of the Zimmer program ([Zim86]) of understanding the higher rank lattice actions on closed manifolds. In the present work, we obtain rigidity results pertaining to this program for weakly hyperbolic actions of higher rank lattices on tori (see e.g. IV.17).

In another direction, one can ask to what extent does the orbit structure of a group action determine the acting group. The following theorem shows that the group structure is completely determined for actions of connected higher real rank semisimple Lie groups:

Theorem I.6 ([Zim80], Zimmer). *Suppose G and G' are connected semisimple Lie*

groups with finite center and no compact factors. Suppose S (respectively S') is an essentially free ergodic irreducible G (respectively G')-space with finite invariant measure, and assume that the actions are orbit equivalent. Assume $\mathbb{R}\text{-rank}(G) \geq 2$.

Then

- G and G' are locally isomorphic
- In the center-free case, $G \cong G'$, and identifying G and G' via this isomorphism, the actions of G on S and S' are isomorphic.

In the remainder of this chapter we will briefly review some of the history and results leading up to the present work. First we will discuss the history related to the ergodic theory of weakly hyperbolic actions. We conclude this chapter with a review of results relating to weakly hyperbolic higher rank lattice actions.

1.2 Ergodicity of Weakly Hyperbolic Actions

Ergodic theory is the study of group actions in the measurable category. A common theme in mathematics is to identify the irreducible structures and to understand how they fit together in the general case. For measurable group actions, the irreducible pieces are the *ergodic* actions (II.1). For a general measure preserving group action, there is the ergodic decomposition (II.3) that decomposes the action as an average of ergodic actions. Consequently, it is natural when studying group actions that preserve a finite measure to determine if they are ergodic, and if not, the nature of their ergodic pieces.

A class of dynamical systems that tend to be ergodic are the partially hyperbolic dynamical systems. Generally, a smooth dynamical system with some exponential expansion or contraction is described as a partially hyperbolic system. A fully hyperbolic system is characterized by having complementary directions that expand

and contract in the phase space.

To get a feeling for why hyperbolicity implies irreducibility, consider the image of a small quadrilateral under an Anosov diffeomorphism on \mathbb{T}^2 defined by a hyperbolic matrix. If the box has one side parallel to the contracting eigenspace and the other side parallel to the expanding eigenspace, then iterates of the diffeomorphism transform this box into a very thin and long strip. As the torus is closed, this strip is forced to wind all around the torus. Consequently, the integer action generated by the Anosov diffeomorphism could not have any interesting invariant subactions.

Though this only gives a feeling for why hyperbolic systems are irreducible, the existing proofs for ergodicity of various partially hyperbolic systems ultimately rely on the existence of expanding and contracting directions as above. Historically, the theory began with theorems of E. Hopf on the ergodicity of geodesic flows in negative curvature:

Theorem I.7 ([Hop39],[Hop40], Hopf). *Let (M, g) be a closed surface of variable negative curvature or a closed manifold of constant negative sectional curvature in any dimension. Then the geodesic flow g_t on the unit tangent bundle UM is ergodic.*

Hopf proved his theorems around 1940. It wasn't until 1967 that general theorems were proven by Anosov and Sinai. They focused on dynamical systems with complementary uniformly expanding and contracting directions, establishing their ergodicity:

Theorem I.8 ([Ano67], [AS67], Anosov and Sinai). *Let (M, g) be a closed manifold with variable negative curvature. Then the geodesic flow g_t on the unit tangent bundle UM is ergodic. More generally, C^2 volume preserving Anosov diffeomorphisms and flows are ergodic.*

Their proof for ergodicity are along the same line as Hopf's, but were much more technical. In particular, they had to establish a regularity property for foliations associated to the contracting and expanding directions now known as *absolute continuity* (II.10).

Since then, it has been shown that Anosov diffeomorphisms and flows satisfy stronger statistical properties than ergodicity such as mixing, the K -property, and the Bernoulli property. In fact, these properties are known to hold for a wider class of non-uniform hyperbolic systems. A comprehensive treatment of results of this sort can be found in ([OW98]).

In view of such strong statistical results for hyperbolic systems, it is natural to try to relax the hyperbolicity and study which statistical properties persist. This is currently an active area of research in the theory of partially hyperbolic dynamics. One way to relax the hyperbolicity condition is to allow other directions that may or may not exhibit exponential expansion or contraction. If this is done in a uniform way, one obtains uniformly partially hyperbolic diffeomorphisms (II.6). In many cases, ergodicity persists. As an illustration there is the following:

Theorem I.9 ([PS00], Pugh and Shub). *Let f be a volume preserving C^2 partially hyperbolic diffeomorphism on a closed manifold M . If f is center bunched, dynamically coherent, and essentially accessible, then f is ergodic.*

For general group actions with hyperbolic behavior, less is known about ergodicity. Pugh and Shub studied the ergodicity of locally free actions of Lie groups G on closed manifolds with some group element that $g \in G$ that is hyperbolic transverse to the orbit foliation. They call such actions *Anosov Actions* and proved the following theorem:

Theorem I.10 ([PS72], Pugh and Shub). *Suppose $\phi : G \rightarrow \text{Diff}^2(M)$ is a volume preserving Anosov action with an Anosov element in the centralizer of G . Then ϕ is ergodic.*

In the present work, we will establish the ergodicity of C^2 volume preserving weakly hyperbolic actions of discrete groups on closed manifolds (III.14). As a simple corollary (III.15), we obtain that weakly hyperbolic actions are *weakly mixing* (II.5). In particular, we recover the ergodicity and weak mixing of C^2 volume preserving Anosov diffeomorphisms as a corollary. The proof ultimately relies on Hopf's original argument.

1.3 Higher Rank Lattice Actions

A prototypical example of a higher rank lattice (II.23) is the group $SL(n, \mathbb{Z})$ for $n \geq 3$. These groups have a number of remarkable rigidity properties. The Zimmer program ([Zim86]) seeks to understand the volume preserving actions of a higher rank lattice Γ on closed manifolds. Examples of such actions are essentially derived from the so called standard actions.

The first type of standard action are the actions of higher rank lattices by isometries of a Riemannian metric. These actions come from morphisms of Γ into compact groups. These actions exhibit no hyperbolic behavior. The second type of standard actions are given by actions on a nilmanifold N/Λ (II.19) defined by a morphism of Γ into $\text{Aut}(N/\Lambda)$. These actions generalize the action of $SL(n, \mathbb{Z})$ on the n -torus by automorphisms. The third type of standard actions are given by actions on compact homogeneous spaces H/Λ by left translation through a morphism $\Gamma \rightarrow H$.

The second and third type actions fit into the larger class of actions by affine diffeomorphisms on homogeneous spaces. These actions do exhibit some hyperbolicity.

For an algebraic characterization of weak hyperbolicity for the standard actions see Remark III.3. In the present work, we will investigate general volume preserving weakly hyperbolic actions of higher rank lattices on tori. Under suitable hypotheses, we obtain that all such actions are semiconjugated to a weakly hyperbolic action by automorphisms (IV.17).

One aspect of the Zimmer program is to show that there are no non-trivial perturbations of the standard actions. An action ϕ is said to be *locally rigid* if any action ϕ' which is close to ϕ in the space of actions is conjugated back to ϕ by a small conjugating map. For more precise definitions and an excellent survey we refer the reader to ([Fis]).

Since the inception of Zimmer's program, there have been numerous results that exploit some form of hyperbolicity. Perhaps the first such result was due to Hurder, where he used the presence of many Anosov diffeomorphisms to prove the following:

Theorem I.11 ([Hur92], Hurder). *The standard action of a finite index subgroup of $SL(n, \mathbb{Z})$ on \mathbb{T}^n is deformation rigid when $n \geq 3$.*

Here, the word *deformation* refers to the fact that the nearby actions are connected to the original action by a continuous path of actions. Following Hurder's work, the next foundational result on the local rigidity of standard higher rank lattice actions was given in the next theorem due to Katok and Lewis.

Theorem I.12 ([KL91], Katok and Lewis). *Let Γ be a finite index subgroup of $SL(n, \mathbb{Z})$, $n > 3$. Then the standard action of Γ on \mathbb{T}^n is locally rigid.*

As in Hurder's work, the presence of Anosov diffeomorphisms and their stability plays an important role in the proof of I.12. While there were further developments concerning the local rigidity of the standard *Anosov* higher rank lattice actions on

nilmanifolds, the next break through on local rigidity for higher rank lattice actions without assuming the presence of group elements acting by Anosov diffeomorphisms was proven by Margulis and Qian in [MQ01]. Therein they define the *weakly hyperbolic* actions and prove the following:

Theorem I.13 ([MQ01], Margulis and Qian). *Let H be a real algebraic Lie group and Λ a cocompact lattice in H . Let Γ be a higher rank lattice. Then any standard weakly hyperbolic affine action of Γ on H/Λ is locally rigid.*

We remark that since the work of Margulis and Qian, there are more general local rigidity results due to Fisher and Margulis ([FM]). The techniques of Margulis and Qian are widely applicable. In addition to I.13 they proved a global rigidity result for Anosov actions of higher rank lattices on nilmanifolds, generalizing earlier work in articles by Katok, Lewis, Qian, Spatzier, and Zimmer.

Theorem I.14 ([MQ01], Margulis and Qian). *Let ρ be an Anosov action of a higher rank lattice Γ on a nilmanifold. Suppose ρ preserves a full support invariant measure and is covered by an action on the universal cover. Then after passing to a finite index subgroup Γ' , ρ is continuously conjugated to an action by automorphisms.*

In the present work, we obtain results that generalize I.14 to the class of weakly hyperbolic actions of higher rank lattices on tori (see e.g. IV.21). Along the way, we show that for such actions, weak hyperbolicity persists in the induced action on first homology, answering a question posed in [MQ01].

CHAPTER II

Preliminaries

This chapter collects together preliminary material that is referred to in subsequent chapters.

2.1 Ergodic Actions and Ergodic Decomposition

Ergodic theory is the study of measurable group actions. We will be considering measurable actions on locally compact second countable metrizable spaces B with a finite measure μ defined on the σ -algebra \mathcal{B} of Borel sets. Once the measure is normalized so that $\mu(B) = 1$, such spaces are called *standard Borel probability spaces*. The *support* of the measure μ is defined to be the complement of the largest open set of B with zero μ -measure. A Borel set $E \subset B$ is *conull* if its complement E^C is a μ -null set. A map

$$T : (B, \mu) \rightarrow (B, \mu)$$

is said to be a *Borel map* if the preimage of a Borel set is a Borel set. Given a Borel map T , there is a push-forward Borel measure $T_*\mu$ on B defined by

$$T_*\mu(E) := \mu(T^{-1}(E)),$$

for each Borel set $E \in \mathcal{B}$.

Let Γ denote a group and (B, μ) a standard Borel probability space. A group action of Γ on (B, μ) by Borel maps is said to be *measure preserving* if $\gamma_*\mu = \mu$ for each $\gamma \in \Gamma$. Given a measure preserving Γ -action

$$\phi : \Gamma \rightarrow \text{Aut}(B, \mu),$$

precomposition by the action induces a unitary representation λ_ϕ on $L^2(B, \mu)$. Specifically,

$$\lambda_\phi : \Gamma \rightarrow \mathcal{U}(L^2(B, \mu)),$$

is defined by $\lambda_\phi(\gamma) \cdot f := f \circ (\gamma^{-1})$ for each $\gamma \in \Gamma$ and $f \in L^2(B, \mu)$.

Definition II.1. A Γ -action ϕ on a probability space (B, μ) is said to be *ergodic* if for each $\phi(\Gamma)$ -invariant Borel set $E \in \mathcal{B}$,

$$\mu(E) \cdot \mu(E^C) = 0.$$

To prove that an action is ergodic, it is useful to have a different formulation of ergodicity (see e.g. [Wal75, Theorem 1.6]).

Theorem II.2. *Let ϕ be a Γ action on a probability space (B, μ) . Then ϕ is ergodic if and only if all fixed vectors of the associated unitary representation λ_ϕ belong to the subspace of (essentially) constant functions.*

Ergodicity of a group action may be thought of as a measure theoretic formulation of irreducibility. When a locally compact second countable group G acts measure-preservingly on a Borel probability space (B, μ) , the measure μ decomposes as an average of ergodic invariant measures, the supports of which partition B modulo μ -null sets (see e.g. [Fer80, Theorem 2.3.2]). Briefly, this follows from the fact that the space of G -invariant measures on B is a compact convex set in a linear space with extreme points given by the ergodic measures. One then applies Choquet's

theorem from convex analysis which asserts that points in a compact convex set are expressible as an average of extreme points. For the present work, we only need the ergodic decomposition for a single transformation and particularly like the following formulation:

Theorem II.3 (Theorem 2.19 in [DF04]). *Let T be a measure preserving transformation of a Borel probability space (B, μ) . Then there exist a conull T -invariant Borel set $B' \subset B$, a standard Borel probability space (Ω, ν) , a Borel map*

$$\xi : \Omega \rightarrow \text{Prob}(B'),$$

and a T -invariant Borel map

$$\psi : B' \rightarrow \Omega$$

such that:

- $\xi(\omega)(\psi^{-1}(\omega)) = 1$ for each $\omega \in \Omega$,
- $\mu = \int_{\Omega} \xi(\omega) d\nu(\omega)$, and
- $\xi(\omega)$ is quasi-invariant and ergodic for each $\omega \in \Omega$.

Now suppose that a single transformation T acts on a probability space (B, μ) preserving the measure. Given a function f on B , one can try to compute the asymptotic average value of f along the orbits of T . It turns out that if f is integrable, this time average converges in a conull subset of B . Moreover, if the system (T, B, μ) is ergodic, then the time average of f , an invariant function, will agree with the space average of f almost everywhere.

Theorem II.4 (Birkhoff Ergodic Theorem). *Let T be a measure preserving invertible transformation of a probability space (B, μ) and let $f \in L^1(B, \mu)$. Then for μ -almost*

every $x \in B$, the following time averages exist and agree:

$$f^+(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)),$$

$$f^-(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^{-i}(x)).$$

Moreover, if (T, B, μ) is ergodic, then

$$f^+(x) = f^-(x) = \int_B f d\mu$$

for μ -almost every $x \in B$.

We conclude this section by defining a statistical property for measurable group actions that is stronger (implies) ergodicity of the action.

Definition II.5. A Γ -action ϕ on a probability space (B, μ) is said to be *weakly mixing* if the diagonal action $\phi \times \phi$ of Γ on $(B \times B, \mu \times \mu)$ is ergodic.

For an excellent survey on the weak mixing property for group actions, we refer the reader to [BG04].

2.2 Partially Hyperbolic Diffeomorphisms and Associated Foliations

Let M^n be a closed orientable Riemannian n -manifold and let μ denote a smooth probability measure on M^n (i.e. obtained by integrating a smooth nondegenerate differential n -form on M). Let $\text{Diff}_\mu^k(M)$ denote the group of μ -preserving C^k -diffeomorphisms on M . Smooth hyperbolic dynamics is the study of group actions that infinitesimally exponentially expand or contract distances in M . The conclusions one can draw depend on the degree of uniformity of expansion or contraction in M . In the present work, we will be considering a well studied class of diffeomorphisms that exhibit uniform expansion and contraction in some directions at rates bounded away from the rates of expansion and contraction in the remaining directions.

Definition II.6. An element $f \in \text{Diff}^k(M)$ ($k \geq 1$) is a *partially hyperbolic diffeomorphism* if there exist continuous df -invariant subbundles $E_f^s, E_f^c, E_f^u \subset TM$, and real numbers $C \geq 1, a > b \geq 1$ such that:

$$TM = E_f^s \oplus E_f^c \oplus E_f^u,$$

and for all $v^u \in E_f^u, v^c \in E_f^c, v^s \in E_f^s$ and for all positive integers n ,

- $\|d(f^n)v^u\| \geq C^{-1}a^n\|v^u\|$
- $\|d(f^n)v^s\| \leq Ca^{-n}\|v^s\|$
- $C^{-1}b^{-n}\|v^c\| \leq \|d(f^n)v^c\| \leq Cb^n\|v^c\|$.

We denote the set of volume preserving partially hyperbolic C^k -diffeomorphisms by $PH_\mu^k(M)$. The invariant distributions E_f^s and E_f^u are called the *stable* and *unstable distributions*.

Definition II.7. A d -dimensional continuous foliation \mathcal{W} of M^n by C^k -leaves is a partition of M into immersed d -dimensional C^k -submanifolds called *leaves* so that each point $x \in M$ has a *foliated neighborhood*, i.e a map

$$\Gamma : B^d \times B^{n-d} \rightarrow M$$

where B^d denotes the ball of dimension d such that

- Γ is a homeomorphism onto an open set in M taking $(0,0)$ to x ,
- for each $y \in B^{n-d}$, the map

$$\Gamma(\cdot, y) : B^d \rightarrow M$$

belongs to $C^k(B^d, M)$ and locally defines a leaf of the foliation, and

- the map

$$B^{n-d} \rightarrow C^k(B^d, M)$$

given by $y \mapsto \Gamma(\cdot, y)$ is continuous in the C^k -topology.

The leaf through the point x is denoted by $\mathcal{W}(x)$.

Theorem II.8 (Theorem 5.5 in [MH77] , [Pes73]). *Let f be a partially hyperbolic diffeomorphism. Then the stable and unstable distributions uniquely integrate to C^k -immersed submanifolds, forming continuous f -invariant foliations of M .*

These foliations are denoted by \mathcal{W}_f^s and \mathcal{W}_f^u and are called the *stable and unstable foliations*.

As the length of vectors in E_f^s and E_f^u contract (respectively expand) exponentially, one should expect that distances in leaves of \mathcal{W}_f^s and \mathcal{W}_f^u contract (respectively expand) exponentially. In fact, the leaves of these foliations are characterized dynamically:

For sufficiently small $\delta > 0$ there is a constant $K(\delta) > 0$ such that

$$\mathcal{W}_f^s = \{y \in M : d(f^i(x), f^i(y)) < K(a^{-1} + \delta)^i d(x, y), \forall i \in \mathbb{N}\}$$

and

$$\mathcal{W}_f^u = \{y \in M : d(f^{-i}(x), f^{-i}(y)) < K(a - \delta)^{-i} d(x, y), \forall i \in \mathbb{N}\}.$$

2.3 Absolutely Continuous Foliations

In Chapter 3, we will consider the ergodic theory of volume preserving actions of groups with elements that act by partially hyperbolic diffeomorphisms on a closed manifold M . Consequently, it is important to consider the relationship between the (un)stable foliations and the volume on M .

Definition II.9. Let \mathcal{W} be a continuous foliation of M by C^k -leaves. A smoothly immersed submanifold $T \subset M$ is a *transversal* to \mathcal{W} if

$$T_x(M) = T_x(\mathcal{W}(x)) \oplus T_x(T)$$

for each $x \in T$.

Let \mathcal{W} be a continuous foliation of M by submanifolds with some given regularity. In order to mimic the classical Fubini theorem, one would like that the volume of a Borel set $E \subset M$ is obtained by integrating the volume of E in leaves along a transversal to the foliation.

Definition II.10. A foliation \mathcal{W} of M is said to be *absolutely continuous* if for each open set U of M which is a union of local leaves and each transversal T to the foliation there is a measurable family of positive measurable functions

$$\{\delta_x : \mathcal{W}(x) \cap U \rightarrow \mathbb{R}\}_{x \in T}$$

so that for each measurable subset $E \subset U$,

$$\mu(E) = \int_T \int_{\mathcal{W}(x) \cap U} \chi_E(x, y) \delta_x(y) d\mu_{\mathcal{W}(x)}(y) d\mu_T(x),$$

where $\mu_{\mathcal{W}(x)}$ and μ_T denote the Riemannian volumes in $\mathcal{W}(x)$ and T respectively.

Absolute continuity of a foliation as formulated above implies that zero volume subsets in the foliated manifold have zero volume in leaves through almost all points.

Lemma II.11 (Lemma 5.4 in [Bri95]). *Let \mathcal{W} be an absolutely continuous foliation of a manifold M and let $N \subset M$ be a null set. Then there is a null set $N_1 \subset M$ such that for any $x \in M - N_1$ the intersection $\mathcal{W}(x) \cap N$ has conditional measure zero in $\mathcal{W}(x)$.*

Surprisingly, there are examples of dynamically defined continuous foliations that are not absolutely continuous ([Mil97]). A strictly stronger notion ([Bri95, Proposition 3.5]) is that of *transversal absolute continuity* of a foliation. To define this, first note that for any points $x_1 \in M$ and $x_2 \in \mathcal{W}(x_1)$ and choice of transversals T_i to the foliation through the points x_i ($i = 1, 2$), there is an associated *Poincare map*. This map is a homeomorphism

$$p : U_1 \rightarrow U_2$$

between neighborhoods U_i of x_i in T_i satisfying $p(x_1) = x_2$ and $p(x) \in \mathcal{W}(x)$ for each $x \in U_1$.

Definition II.12. A foliation \mathcal{W} is *transversally absolutely continuous* if all its Poincare maps are absolutely continuous maps with respect to the induced Riemannian measures on the transversals. In other words, for each choice of transversals L_1 and L_2 and associated Poincare map p , there is a positive measurable Jacobian

$$J : U_1 \rightarrow \mathbb{R}$$

such that for each measurable subset $A \subset U_1$,

$$\mu_{T_2}(p(A)) = \int_{U_1} \chi_A(x) J(x) d\mu_{T_1}(x).$$

If, in addition these Jacobians are continuous and positive, then the foliation is said to be *measurewise C^1* .

Of technical importance is the following:

Theorem II.13 (Theorem 2.1 in [PS72]). *Let $f \in PH_\mu^k(M)$ and suppose $k \geq 2$. Then the stable and unstable foliations of f are measurewise C^1 .*

In the course of their proof, Pugh and Shub gave an asymptotic expression for the Jacobians of the Poincare maps. Starting from this expression, Nitica and Torok fur-

ther strengthened the absolute continuity property for stable and unstable foliations. They studied the regularity properties of these foliations, first translating Pugh and Shub's work into a statement about the absolute continuity of the local coordinate charts for the foliation. Their formulation shows that Jacobians are differentiable along leaves and is useful for studying the regularity properties of functions on M that restrict to regular functions on the leaves. The following follows from the proof of their regularity theorem:

Theorem II.14 (Theorem 6.4 in [NT98]). *Let $f \in PH_\mu^2(M)$. Then there is a stable foliation chart*

$$\Gamma : B^d \times B^{n-d} \rightarrow M$$

around each $p \in M$ so that

$$\Gamma^*(\mu) = J(x, y) dx dy$$

where the Jacobian J is an everywhere positive and continuous function on $B^d \times B^{n-d}$ that has continuous (in x and y variables) partials of first order in the x variables.

Definition II.15. When a foliation \mathcal{F} has local foliation charts as described in Theorem II.12, the foliation \mathcal{F} is said to be *strongly absolutely continuous*.

2.4 Lie Groups and Lattices

This section contains the necessary results concerning Lie groups and their lattices. Let G be a connected Lie group with Lie algebra \mathfrak{g} . The exponential map $\exp : \mathfrak{g} \rightarrow G$ is a local diffeomorphism in a neighborhood of O of the origin in \mathfrak{g} onto a neighborhood I of the identity in G . Let $\log : I \rightarrow \mathfrak{g}$ denote the inverse. To see what the multiplication map $G \times G \rightarrow G$ looks like in the Lie algebra, define the map

$$C : O \times O \rightarrow O$$

by $C(A, B) = \log(\exp(A), \exp(B))$ for $A, B \in \mathcal{O}$.

Theorem II.16 (Baker-Campbell-Hausdorff Formula). *In the above notation,*

$$C(A, B) = (A + B) + \sum_{i=1}^{\infty} b_i(A, B),$$

where each

$$b_i : \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}$$

is a skew bilinear map defined as a rational linear combination of terms involving i Lie bracket operations.

Definition II.17. Let G be a connected Lie group. A closed subgroup $\Gamma \subset G$ is a *lattice* in G if Γ is discrete and G/Γ supports a finite G -invariant measure. A discrete subgroup Γ is said to be *uniform lattice* if G/Γ is compact.

For example $\mathbb{Z}^n \subset \mathbb{R}^n$ is a uniform lattice since the quotient $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is compact. In fact, for all connected, simply connected nilpotent Lie groups, lattice subgroups are uniform.

Theorem II.18 (Theorem 2.1 in [Rag72], Malcev). *Let N be a connected, simply connected nilpotent Lie group and $\Lambda \subset N$ a discrete subgroup. Then Λ is a lattice subgroup if and only if N/Λ is compact.*

Definition II.19. Compact manifolds of the form N/Λ , with N a simply connected nilpotent Lie group and Λ are called *nilmanifolds*.

Remark II.20. It follows from basic theory that $\pi_1(N/\Lambda) = \Lambda$ for a nilmanifold.

Linear algebra shows that any automorphism of \mathbb{Z}^n extends to a unique automorphism of \mathbb{R}^n . This fact is also true for lattices in connected, simply connected nilpotent Lie groups.

Theorem II.21 (Theorem 2.11 in [Rag72], Malcev). *Let N be a connected, simply connected nilpotent Lie group and Λ a lattice in N . Then any automorphism of Λ extends uniquely to an automorphism of N .*

Remark II.22. Theorem II.21 furnishes an identification $\text{Aut}(N/\Lambda) \cong \text{Aut}(\Lambda) \subset \text{Aut}(N)$.

In Chapter 4, we will analyze actions of lattices on nilmanifolds. The acting lattices will be a class of lattices in semisimple Lie groups known for their exceptional rigidity properties. These lattices are defined in the next definition.

Definition II.23. A *higher rank lattice* is a lattice subgroup of a connected semisimple Lie group with finite center and noncompact simple direct factors each of which has real rank at least two.

Higher rank lattices have a rich structure and many wonderful properties. A detailed account of higher rank lattices may be found in [Mar91]. One such property that is used in the present work is the content of the following:

Theorem II.24 (Proposition IX 5.7 in [Mar91]). *Let Γ be a higher rank lattice and*

$$\pi : \Gamma \rightarrow \text{GL}(V)$$

a finite dimensional representation of Γ over a local field of characteristic zero. Then the Zariski closure of $\pi(\Gamma) \subset \text{GL}(V)$ is semisimple.

Recall that a finite dimensional representation π of a group Γ is said to be *completely reducible* if each subrepresentation of π has a complementary subrepresentation. The importance of Theorem II.24 for the present work comes from the fact that the finite dimensional representations of semisimple Lie groups are completely reducible.

Corollary II.25. *Let Γ be a higher rank lattice and π a finite dimensional representation of Γ . Then π is completely reducible.*

2.5 Group Cohomology

In Chapter 4 we will use several cohomological vanishing result that we summarize in this section. First we recall the definition of group cohomology with coefficients in a representation. A good general reference on the cohomology of groups is [Bro80].

Let G be a group and π a representation of G on a vector space V . Let $\mathbb{Z}G$ denote the integral group ring and consider \mathbb{Z} as a trivial $\mathbb{Z}G$ module. Take a projective resolution (P_i, δ_i) of $\mathbb{Z}G$ by $\mathbb{Z}G$ modules. This resolution gives rise to a cochain complex $(\text{Hom}_{\mathbb{Z}G}(P_i, V), \delta^i)$, where

$$(\delta^i \phi)(x) = (-1)^{i+1} \phi(\delta_{i+1} x)$$

for each $\phi \in \text{Hom}_{\mathbb{Z}G}(P_i, V)$ and $x \in P_{i+1}$. It turns out that the cohomology groups of this complex are independent of the choice of resolution. They are denoted by $H^*(G, \pi)$ and are called the cohomology groups of G with coefficients in π .

A well known resolution, the bar resolution, leads to an explicit description of the first cohomology group $H^1(G, \pi)$.

Definition II.26. A *1-cocycle* is a map $c : G \rightarrow V$ that satisfies the cocycle identity

$$c(gh) = c(g) + \phi(g)(c(h))$$

for each $g, h \in G$. A *coboundary* is a cocycle c_v of the form

$$c_v(g) := v - \pi(g)(v)$$

for some fixed $v \in V$. The first cohomology of G with coefficients in the representation π is the group of cocycles $Z^1(\Gamma, \pi)$ modulo the normal subgroup of coboundaries $B^1(\Gamma, \pi)$.

As a consequence of his arithmeticity theorem for higher rank lattices, Margulis proved that their first cohomology with coefficients in a finite dimensional representation over a local field of characteristic zero vanish. The next theorem is a generalization due to Starkov.

Theorem II.27 ([Sta02], Starkov). *Let G be a connected semisimple Lie group with a lattice Γ . Then either there is no epimorphism of G onto a Lie group H locally isomorphic to $SO(1, n)$ or $SU(1, n)$ with $\phi(\Gamma)$ a lattice in H , or $H^1(\Gamma, \pi) = 0$ for all finite dimensional representations π of Γ over \mathbb{R} .*

Next, we recall a well known relationship between the second cohomology group and splitting short exact sequences (see e.g. [Bro80]).

Theorem II.28. *Suppose that A is an abelian group and that*

$$1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$$

is a short exact sequence of groups. If $H^2(C, A) = 0$, then the sequence splits.

Remark II.29. Let Γ be a group acting on a closed manifold M . Then lifting the group action to the universal cover of M may be formulated in terms of splitting the short exact sequence

$$1 \rightarrow \pi_1(M) \rightarrow L(\Gamma) \rightarrow \Gamma \rightarrow 1$$

where $L(\Gamma)$ is the group of all lifts of all elements of Γ to the universal cover of M .

In addition to the first cohomology vanishing, some higher rank lattices are known to have second cohomology vanishing.

Theorem II.30 ([Rag66], Raghunathan). *Let Γ be a uniform lattice in a connected simple Lie group with real rank at least three. Then $H^2(\Gamma, \pi) = 0$ for all finite dimensional real representations.*

Next we consider groups with vanishing first cohomologies in unitary representations.

Definition II.31. Let Γ be a discrete group and \mathcal{H} a separable Hilbert space. A unitary representation π is said to have *almost invariant vectors* if for every $\epsilon > 0$ and each compact subset $K \subset \Gamma$, there exists a unit vector $u(\epsilon, K) \in H$ so that $\|\pi(\gamma)u - u\| < \epsilon$ for each $\gamma \in K$. The group Γ has *Kazhdan's property (T)* if every unitary representation of Γ with almost invariant vectors has a nonzero invariant vector.

It is well known and essentially due to Kazhdan that higher rank lattices have this fixed point property (see e.g. [dlH89]). There is a useful characterization of this property in terms of the vanishing of the first cohomology group of Γ with coefficients in unitary representations.

Theorem II.32 ([Del77], [Gui80], Delorme and Guichardet). *A group Γ has Kazhdan's property if and only if $H^1(\Gamma, \phi) = 0$ for every unitary representation ϕ of Γ .*

CHAPTER III

Accessibility and Ergodicity of Weakly Hyperbolic Actions

3.1 Introduction

In this chapter we define weakly hyperbolic actions and establish their ergodicity. As a corollary, we obtain that weakly hyperbolic actions are weakly mixing. The proof given for ergodicity is ultimately based on Hopf's ([Hop39],[Hop40]) argument for ergodicity of the geodesic flow on the unit tangent bundle of a hyperbolic surface. Building on Hopf's argument, Anosov and Sinai proved the ergodicity of Anosov diffeomorphisms and flows in [Ano67] and [AS67]. To illustrate the main points of Hopf's argument, we first outline the proof that Anosov diffeomorphisms (I.3) with enough regularity (e.g. C^2 and volume preserving) are ergodic.

Recall that to prove a volume preserving diffeomorphism T of a closed manifold M is ergodic, one must show that a square integrable T -invariant function is almost everywhere constant (II.2). The first step in Hopf's argument is to replace square integrable functions with continuous functions. More precisely, Birkhoff's theorem (II.4) provides continuous maps $f \mapsto f^+$ and $f \mapsto f^-$ of $L^2(M)$ sending a function to its asymptotic average value along forward (respectively backwards) orbits of T . Since the continuous functions are dense in $L^2(M)$ and an invariant function equals its time average, it suffices to prove that the time average of a continuous function

is essentially constant.

The argument proceeds by showing that forward (respectively backward) time averages of continuous functions are constant along stable (respectively unstable) leaves of T . This step essentially uses that distances in stable (unstable) leaves contract exponentially in forward (backwards) time. Again by Birkhoff, the forward and backward time averages converge and agree in a full volume subset $G \subset M$. The argument continues using the fact that in local neighborhoods, any two close points x and y are joined by stable and unstable leaves intersecting in a point z . With control theory in mind, these neighborhoods are commonly referred to as local *accessible* neighborhoods. If the intersection point z belongs to G , it follows that the time average of the function f agree at the points x and y .

Now comes the technical aspect of the argument. To show that the time average of the function f is almost everywhere constant in M , one needs to show that in each local accessible neighborhood as above, the set of points accessed with intersection points in G is conull. Intuitively, one needs Fubini's theorem for the stable and unstable foliations. Provided that the Anosov diffeomorphism is regular enough, these foliations are absolutely continuous (II.10), and the argument may be completed. It is exactly this property that allows one to promote almost everywhere statements with respect to leaves to almost everywhere statements for the foliated manifold M .

Weakly hyperbolic group actions on a manifold M , like Anosov diffeomorphisms, come equipped with (un)stable foliations that exhaust directions in the tangent bundle of M . However, for weakly hyperbolic actions, these foliations are spread amongst finitely many different acting group elements. Therefore, one may not run Hopf's argument for ergodicity as outlined – there is no well defined forward time average for a function. Nevertheless, Hopf's argument may be adapted to show that the

ergodic components of a diffeomorphism with an (un)stable foliation are essentially saturated by (un)stable leaves. This observation together with the regularity Theorem III.10 below is enough to deduce ergodicity for weakly hyperbolic actions as will be illustrated below.

3.2 Weakly Hyperbolic Actions and Local Accessibility

The purpose of this section is to define the actions under consideration as well as the associated notion of local accessibility.

Definition III.1. A *weakly hyperbolic family on M* is a finite subset of $PH_\mu^1(M)$ which is infinitesimally accessible in stable directions. More precisely, it is a finite family $\{\gamma_1, \dots, \gamma_k\} \subset PH_\mu^1(M)$ with associated continuous splittings

$$TM = E_i^s \oplus E_i^c \oplus E_i^u$$

such that $TM = \sum_{i=1}^k E_i^s$. An action of a discrete group Γ on M is *weakly hyperbolic* if there is a finite family of group elements that act as a weakly hyperbolic family on M .

Remark III.2. Weak hyperbolicity for a group action is one possible generalization of the Anosov property for diffeomorphisms. In fact, the weakly hyperbolic actions of the integers are precisely those generated by an Anosov diffeomorphism. It follows easily from the definition that the diagonal product of a weakly hyperbolic action is still weakly hyperbolic.

Example III.3. Weak hyperbolicity for the standard affine actions is characterized algebraically in [MQ01]. First suppose that ρ_A is an action of a higher rank lattice Γ by automorphisms of a nilmanifold given by a homomorphism $\pi : \Gamma \rightarrow \text{Aut}(N/\Lambda)$. Then weak hyperbolicity of ρ_A is equivalent to the induced representation $d(\pi) :$

$\Gamma \rightarrow \text{Aut}(\mathfrak{n})$ not having a nontrivial invariant subspace $\mathfrak{n}_0 \subset \mathfrak{n}$ for which $d(\pi)|_{\mathfrak{n}_0}(\Gamma)$ is precompact in $\text{GL}(\mathfrak{n}_0)$. Next suppose that ρ_T is an action of a higher rank lattice $\Gamma \subset G$ by left translations on a homogeneous space H/Λ given by a homomorphism $\pi : G \rightarrow H$. Then weak hyperbolicity of ρ_T is equivalent to the centralizer of $\pi(G)$ in H being discrete.

The existence of the stable foliations (II.8) suggests that infinitesimal accessibility along stable directions integrates to a notion of local accessibility along stable leaves. This is made precise in the next definition and proposition.

Definition III.4. Let $F = \{\gamma_1, \dots, \gamma_k\}$ be a finite family of partially hyperbolic diffeomorphisms on M with associated stable foliations $\{\mathcal{W}_1^s, \dots, \mathcal{W}_k^s\}$. An *admissible path* for F is a path in M that is piecewise sequentially contained in leaves of the stable foliations: i.e. a path $a : [0, T] \rightarrow M$ with a subdivision $t_0 = 0 \leq t_1 \leq \dots \leq t_k = T$ such that $a([t_{j-1}, t_j]) \subset \mathcal{W}_j^s(a(t_{j-1}))$ for each $j \in \{1, \dots, k\}$. The family F is *locally accessible* if for each $x \in M$ there is an open neighborhood U_x of x such that each $y \in U_x$ is the endpoint of an admissible path for F beginning at x .

Proposition III.5. *A weakly hyperbolic family F of partially hyperbolic diffeomorphisms on a closed manifold M has the local accessibility property.*

Proof. Let $F = \{\gamma_1, \dots, \gamma_k\}$ be a weakly hyperbolic family on M and fix $x \in M$. First note that any piecewise differentiable path in M beginning at x which is piecewise sequentially tangent to the stable distributions $\{E_i\}_{i=1}^k$ is an admissible path for F . This is the case since the stable distributions integrate uniquely (II.8).

As accessibility is a local question, we may view each stable distribution E_i near x as being given by the span of a finite ordered family of continuous vector fields $\{X_i^1, \dots, X_i^{d(i)}\}$, where $d(i) = \dim(E_i)$. Any curve in M which sequentially

defines a solution to the vector fields $\{X_i^j\}_{j=1}^{d(i)}$ is tangent to E_i . Therefore, any curve that sequentially defines a solution to the combined ordered family of fields $X = \{X_1^1, \dots, X_1^{d(1)}, \dots, X_k^1, \dots, X_k^{d(k)}\}$ is admissible for the family F .

The hypothesis that the family F of diffeomorphisms is weakly hyperbolic implies that the family X of vector fields locally spans the tangent bundle near the point x . Establishing local accessibility for a weakly hyperbolic family therefore reduces to the next lemma. \square

Lemma III.6 (Corollary 4.5 in [KP94]). *Let $\{X_1, \dots, X_N\}$ be a family of nonvanishing continuous vector fields in \mathbb{R}^n such that $\text{span}(X_1, \dots, X_N) = \mathbb{R}^n$. Then the set of endpoints of curves beginning at 0 which sequentially define solutions to these fields contains an open set around 0.* \square

Remark III.7. Once continuous local accessibility has been established for a finite family of foliations of a closed manifold M , one may ask finer questions about the set of points accessible from a given point. Suppose, for example, that one is given a full volume subset $G \subset M$ which intersects almost all leaves of the finite family of foliations in leafwise conull sets. Given an initial point $x \in M$ is there an open set U_x of points almost all of which are accessible by a sequence of leaves intersecting in points of G ? A positive answer to this question would provide a key step to generalizing the well known Livsic Theorem ([Liv72]) for bootstrapping the regularity of solutions to cohomology equations over hyperbolic systems to the setting of weakly hyperbolic actions.

3.3 Rauch and Taylor's Theorem

This section presents a regularity result of Rauch and Taylor. It is used in subsequent sections to reduce proving continuity of a function on a manifold to proving

that the function is Lipschitz when restricted to the stable leaves of a family of partially hyperbolic diffeomorphisms. Related regularity results include [HK90], [Jou88], [dLL01], and [RdlL86].

Let \mathcal{F} denote a strongly absolutely continuous foliation and $p \in (1, \infty)$.

Definition III.8. The space $H_{\mathcal{F}}^{1,p}(M)$ consists of functions u on M satisfying

$$\partial_{x_i}(u \circ \Gamma) \in L^p(B^d \times B^{n-d}), i = 1, \dots, d,$$

for each foliation chart Γ as described by Theorem II.13, where differentiation is in the *distributional* sense (see [Zim90] for basics on distributions).

Remark III.9. It is basic to check that the above definition is independent of the choice of chart.

Recently, Rauch and Taylor have established that functions in $H_{\mathcal{F}}^{1,p}(M)$ are microlocally in the standard Sobolev space $H^{1,p}(M)$ away from the conormal bundle of \mathcal{F} ([RT05, Theorem 1.2]). This implies the following:

Theorem III.10 (Theorem 1.1 in [RT05]). *Let $\mathcal{F}_1, \dots, \mathcal{F}_N$ be strongly absolutely continuous foliations of M . Assume that for each $x \in M$,*

$$T_x M = \sum_{j=1}^N T_x \mathcal{F}_j.$$

Then, given $p \in (1, \infty)$, if $u \in H_{\mathcal{F}_j}^{1,p}(M)$ for each $j \in \{1, \dots, N\}$, then $u \in H^{1,p}(M)$, the standard Sobolev space.

We conclude this section by proving a corollary of III.10 outlined to us by Jeffrey Rauch. This corollary is the heart of the regularity bootstrapping arguments used in subsequent sections.

Corollary III.11. *Let $\mathcal{F}_1, \dots, \mathcal{F}_N$ be strongly absolutely continuous foliations of M . Assume that for each $x \in M$,*

$$T_x M = \sum_{j=1}^N T_x \mathcal{F}_j.$$

Suppose that $u \in L^2(M)$ and that there is a constant $K > 0$ so that the restrictions of u to almost all leaves of the foliations $\{\mathcal{F}_j\}$ are almost everywhere K -Lipshitz. Then u agrees almost everywhere with a continuous function.

Proof. As a first step we argue that u has bounded tangential first order derivatives parallel to the leaves of the foliations. These derivatives are only taken in the distributional sense. In a second step, we will use regularity bootstrapping techniques to finish the proof. To this end, fix $j \in \{1, \dots, N\}$, $p \in M$, and let $d = \dim \mathcal{F}_j$. As the regularity properties of u are a local matter, we may identify a local parameterization of \mathcal{F}_j in a neighborhood of $p \in M$ with a foliated open neighborhood $\Omega \subset \mathbb{R}^n$, described by a parameterization

$$\Gamma : U \times V \rightarrow \Omega,$$

with $U \subset \mathbb{R}^d, V \subset \mathbb{R}^{n-d}$ and Jacobian $J(x, y)$ as in II.14.

Let X be a vector field in Ω such that $X \circ \Gamma = \sum_{j=1}^d a_j(x, y) \partial_{x_j}$, with $\partial_{x_i} a_j(x, y) \in C^0(U \times V)$ for $i = 1, \dots, d$ (such a field is well defined, independent of the choice of “flattening” parameterization). Let $U' \subset\subset U$, $V' \subset\subset V$, denote by $\mathcal{D}'(U' \times V')$ the space of distributions on $U' \times V'$, and define

$$X : L_{loc}^1(U' \times V') \rightarrow \mathcal{D}'(U' \times V')$$

by

$$X(f)(\phi) = \int_{V'} \int_{U'} f \left(- \sum_{j=1}^d \partial_{x_j}(\phi a_j) \right) dx dy$$

for $f \in L^1_{loc}(U' \times V')$ and $\phi \in C_c^\infty(U' \times V')$.

Since M has finite volume, $u \in L^1(M)$, and by continuity and positivity of the Jacobian J , it follows that $u \circ \Gamma \in L^1_{loc}(U' \times V')$. To complete this step, we must show that $X(u \circ \Gamma) \in L^\infty(U' \times V') \subset \mathcal{D}'(U' \times V')$. To this end we define (when n is sufficiently large) the difference quotients $u_{n,j}(x, y) := n(u \circ \Gamma(x + \frac{1}{n}e_j, y) - u \circ \Gamma(x, y))$ and the functions $g_n := \sum_{j=1}^d u_{n,j}a_j$. By the hypothesis on u and the continuity of the a_j , each $g_n \in L^\infty(U' \times V')$ and there is a uniform $C > 0$ so that $\|g_n\|_\infty < C$. In particular there is a uniform bound C' on the L^2 -norms of the g_n , so that a subsequence of the g_n weakly converge to a function $g \in L^2(U' \times V')$. This weak limit is also a distributional limit and furthermore it follows from the uniform bound $\|g_n\|_\infty < C$ that $\|g\|_\infty < C$. On the other hand, a routine change of variables argument shows that the g_n converges to $X(u \circ \Gamma)$ in $\mathcal{D}'(U' \times V')$, completing this step.

By the first step and since $u \in L^2(M)$, $u \in H_{\mathcal{F}_j}^{1,2}(M^n)$ for each $j \in \{1, \dots, N\}$. Applying Theorem III.10 establishes that $u \in H^{1,2}(M^n)$. By the Sobolev imbedding theorem (see [Tay96, Proposition 2.2]), $H^{1,p}(M^n) \subset L^{\frac{np}{n-p}}(M^n)$ for $p \in [1, n)$. From this, it follows that $u \in L^{\frac{2n}{n-2}}(M^n)$. Again by the first step, we now obtain $u \in H_{\mathcal{F}_j}^{1, \frac{2n}{n-2}}(M^n)$ for each j and hence $u \in H^{1, \frac{2n}{n-2}}(M^n)$ by III.10.

Note that for all $0 < y < n$,

$$f(y) := \frac{ny}{n-y} - y$$

is positive and increasing as a function of y . We may therefore define a finite increasing sequence $\{x_0, \dots, x_J\}$ inductively by letting $x_0 = 2$ and $x_i = \frac{nx_{i-1}}{n-x_{i-1}}$ whenever $x_{i-1} < n$. Repeating the above argument shows that $u \in H^{1,x_J}(M^n)$. When $x_J = n$, note that $u \in H^{1,n-\epsilon}(M)$ for small $\epsilon > 0$. Choose $\epsilon < \frac{n}{2}$ so that $\frac{n(n-\epsilon)}{n-(n-\epsilon)} > n$. Then $u \in H^{1, \frac{n(n-\epsilon)}{n-(n-\epsilon)}}(M)$. Therefore, $u \in H^{1,p}(M^n)$ for some $p > n$ whether $x_J > n$

or $x_J = n$. The conclusion of the corollary follows by the Sobolev imbedding (see Proposition 2.4 in [Tay96]) $H^{1,p}(M^n) \subset C^0(M^n)$ for $p > n$. \square

3.4 Ergodicity of Weakly Hyperbolic Actions

In this section we use local accessibility together with Corollary III.11 in order to prove that C^2 weakly hyperbolic volume preserving actions of discrete groups are ergodic. Before proceeding, we outline the argument.

Let Γ be a discrete group. Proving ergodicity for a measure preserving Γ action on a Borel probability space (X, μ) is equivalent to showing that any square integrable function which is almost everywhere invariant under the action of Γ is almost everywhere constant (II.2). To accomplish this, we first show that a square integrable function almost everywhere invariant under a partially hyperbolic diffeomorphism $\gamma \in PH_\mu^2(M)$ has the needed tangential regularity in order to apply Corollary III.11. In fact, such a function will be essentially constant on almost all leaves of \mathcal{W}_γ^s .

Next, the weak hyperbolicity property together with Corollary III.11 implies that each group invariant vector in $L^2(M)$ is necessarily equivalent to a continuous function f which is almost everywhere invariant.

Finally, as f is continuous, it is everywhere invariant and hence constant on stable leaves by the Hopf argument. The local accessibility property implies that f is locally constant and therefore constant on M .

The remainder of the section makes this reasoning precise. Recall that for each volume preserving transformation T of M , μ has an ergodic decomposition (II.3). This decomposition consists of a full volume T -invariant Borel set $M' \subset M$, a standard Borel probability space (Ω, ν) , a Borel map

$$\xi : \Omega \rightarrow \text{Prob}(M'),$$

and a T -invariant Borel map

$$\psi : M' \rightarrow \Omega.$$

These maps satisfy

$$\xi(\omega)(\psi^{-1}(\omega)) = 1$$

for each $\omega \in \Omega$,

$$\mu = \int_{\Omega} \xi(\omega) d\nu(\omega),$$

and $\xi(\omega)$ is quasi-invariant and ergodic for each $\omega \in \Omega$. Here, $\text{Prob}(M')$ inherits its Borel structure from the weak-* topology.

Lemma III.12. *Suppose that $T \in PH_{\mu}^1(M)$ has an absolutely continuous stable foliation and that a square integrable function f is almost everywhere T -invariant. Then for almost all leaves, f restricted to the leaf is almost everywhere constant.*

Proof. For $g \in L^1(M, \mu)$ we denote the set of Birkhoff regular points for g by

$$B_g := \{x \in M' \mid g^+(x) = \int_{M'} g d\xi(\psi(x))\},$$

where $g^+(x)$ is defined in II.4. It follows easily from the description of the ergodic decomposition that these sets are Borel sets and the Birkhoff ergodic theorem (II.4) implies these sets have full volume in M . From the separability of $C^0(M)$, it follows that the set

$$B_0 := \bigcap_{g \in C^0(M)} B_g$$

has full volume in M . By hypothesis there is a full volume T -invariant set $I \subset M$ so that f is T -invariant in I . By absolute continuity of the stable foliation, there is a full volume subset of good points $G \subset (I \cap B_f \cap B_0)$ such that $x \in G$ implies $(I \cap B_f \cap B_0)$ is conull in $\mathcal{W}_T^s(x)$ (II.11).

Fix $x \in G$ and let $y_1, y_2 \in \mathcal{W}_T^s(x) \cap (I \cap B_f \cap B_0)$. We first argue that y_1 and y_2 lie in the same ergodic component, or more precisely, that $\xi(\psi(y_1)) = \xi(\psi(y_2))$. Indeed the Hopf argument shows that for any continuous function g , whenever $g^+(x)$ converges, $g^+(y)$ converges to $g^+(x)$ for all $y \in \mathcal{W}_T^s(x)$. Indeed, given $\epsilon > 0$, there is a $N > 0$ so that $|g(T^n x) - g(T^n y)| < \epsilon$ for all $n > N$ since T contracts distances exponentially between points in stable leaves and by uniform continuity of g . This implies that

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} g(T^i(x)) - \frac{1}{n} \sum_{i=0}^{n-1} g(T^i(y)) \right| < \epsilon$$

for all large enough n .

Since $y_1, y_2 \in B_0$, Theorem II.4 implies that

$$\int_M g d\xi(\psi(y_1)) = g^+(y_1) = g^+(y_2) = \int_M g d\xi(\psi(y_2))$$

for all continuous functions g , whence

$$\xi(\psi(y_1)) = \xi(\psi(y_2)).$$

Since $y_1, y_2 \in (I \cap B_f)$,

$$f(y_1) = f^+(y_1) = \int_{M'} f d\xi(\psi(y_1)) = \int_{M'} f d\xi(\psi(y_2)) = f^+(y_2) = f(y_2).$$

□

From Corollary III.11 and Lemma III.12 we deduce the following:

Corollary III.13. *Suppose that f is a square integrable function on M that is almost everywhere invariant under a weakly hyperbolic C^2 family on M . Then f is almost everywhere equal to a continuous function g .*

The ergodicity of weakly hyperbolic actions is the content of the next theorem:

Theorem III.14. *Let ρ be a C^2 volume preserving weakly hyperbolic action of a discrete group Γ on M . Then the Γ action is ergodic.*

Proof. Let f be any square integrable almost everywhere Γ -invariant function. By Corollary III.11, f is almost everywhere equal to a continuous function g that is almost everywhere Γ -invariant. By continuity of g and since full volume sets are dense, g is everywhere invariant and everywhere equal to its forward time average. Now by Hopf's argument g is constant on all stable leaves of all the elements in the weakly hyperbolic family. By Proposition III.5, this family has the local accessibility property. It follows that g is locally constant and hence constant as M is connected. Finally, since g agrees with f almost everywhere, f is essentially constant as required.

□

It follows immediately from Remark III.2 and Theorem III.14 that weakly hyperbolic actions are weakly mixing (II.5).

Theorem III.15. *Let ρ be a C^2 volume preserving weakly hyperbolic action of a discrete group Γ on M . Then the Γ action is weakly mixing.*

In view of Remark III.2, we recover Anosov's classical result on the ergodicity and weak mixing of Anosov diffeomorphisms.

Corollary III.16 ([Ano67], Anosov). *Let A be a C^2 volume preserving Anosov diffeomorphism on a closed manifold M . Then A is ergodic.*

CHAPTER IV

Weakly Hyperbolic Group Actions on Nilmanifolds

4.1 Introduction

In this chapter we restrict our attention to weakly hyperbolic group actions of large groups on nilmanifolds. The results of this chapter fit into the Zimmer program ([Zim86]) of classifying ergodic volume preserving higher rank lattice actions on closed manifolds. Briefly, it is conjectured that all such actions are built on an open dense set of the manifold from a collection of three types of algebraic actions on algebraic spaces. The three conjectural building block actions consist of isometric actions, actions by left translation on homogenous spaces, and actions by nilmanifold automorphisms. One may break this problem into two complementary subproblems. The first is to identify the closed manifolds supporting higher rank lattice actions. The second is to classify the different possible actions on the manifolds that support actions.

About twenty years prior to the inception of Zimmer's classification program, Smale ([Sma67]) posed the problem of classifying Anosov diffeomorphisms on closed manifolds. Again, one may view Smale's problem as consisting of a manifold recognition problem together with the problem of classifying Anosov diffeomorphisms on the model pieces. For the case of recognition it is still conjecturally open that the

only closed manifolds supporting Anosov diffeomorphisms are finitely covered by nilmanifolds.

The continuous classification problem for volume preserving Anosov diffeomorphism on nilmanifolds was settled by work of Franks and Manning in two complementary theorems. For simplicity, we state the specialization of their theorems to tori.

Theorem IV.1 ([Fra70], Franks). *Let $f \in \text{Diff}_{\text{vol}}(\mathbb{T}^n)$ be an Anosov diffeomorphism, $f_* \in GL(n, \mathbb{Z})$ be the induced automorphism on $H_1(\mathbb{T}^n, \mathbb{Z})$, and f_0 be the automorphism of \mathbb{T}^n induced by f_* . If f_* has no eigenvalues of modulus one, then there is a unique homeomorphism $\phi \in \text{Homeo}(\mathbb{T}^n)$ homotopic to the identity such that $\phi \circ f = f_0 \circ \phi$.*

Theorem IV.2 ([Man74], Manning). *Let $f \in \text{Diff}_{\text{vol}}(\mathbb{T}^n)$ be an Anosov diffeomorphism and $f_* \in GL(n, \mathbb{Z})$ be the induced automorphism on $H_1(\mathbb{T}^n, \mathbb{Z})$. Then f_* has no eigenvalues of modulus one.*

Combining these shows that any volume preserving Anosov diffeomorphism on a torus is, up to conjugation by a unique homeomorphism, a hyperbolic automorphism. In view of Remark III.2, it is natural to ask if a similar continuous classification holds for volume preserving weakly hyperbolic actions on tori. When the acting group is a higher rank lattice, a positive answer provides supporting evidence for Zimmer's program. The main result in this chapter is Theorem IV.12, an analogue to Manning's theorem for weakly hyperbolic actions of property (T) groups on tori. In the final section of this chapter, we address the continuous classification problem for volume preserving weakly hyperbolic higher rank lattice actions on tori.

4.2 Actions Covered by Actions

In this section, we study group actions on nilmanifolds that are covered by an action on the simply connected nilpotent Lie group. The section consists of a series of calculations and will be referred to in subsequent sections.

Throughout, we let Γ denote a discrete group and

$$\rho : \Gamma \rightarrow \text{Diff}^k(N/\Lambda)$$

denote an action of Γ by C^k -diffeomorphisms on a nilmanifold N/Λ (II.19). Moreover, we assume that the action ρ is covered by an action

$$\bar{\rho} : \Gamma \rightarrow \text{Diff}^k(N)$$

on the universal cover.

Lemma IV.3. *There is a homomorphism*

$$\pi : \Gamma \rightarrow \text{Aut}(\Lambda)$$

satisfying

$$\bar{\rho}(\gamma)(n\lambda) = \bar{\rho}(\gamma)(n) \cdot \pi(\gamma)(\lambda)$$

for each $n \in N$, $\lambda \in \Lambda$, and $\gamma \in \Gamma$.

Proof. Fix $\gamma \in \Gamma$. Since $\bar{\rho}$ covers ρ , for each $n \in N$ and $\lambda \in \Lambda$, there is a unique group element $\pi(\gamma)(n, \lambda) \in \Lambda$ such that

$$\bar{\rho}(\gamma)(n\lambda) = \bar{\rho}(\gamma)(n) \cdot \pi(\gamma)(n, \lambda).$$

This defines a map

$$\pi(\gamma) : N \times \Lambda \rightarrow \Lambda$$

which is continuous in the first variable. As N is connected and Λ is discrete, this map is independent of N , giving a map

$$\pi(\gamma) : \Lambda \rightarrow \Lambda,$$

satisfying the required identity.

Next we check that for fixed $\gamma \in \Gamma$, $\pi(\gamma) \in \text{Hom}(\Lambda)$. Let $\lambda_1, \lambda_2 \in \Lambda$ and compute that

$$\begin{aligned} \bar{\rho}(\gamma)(n) \cdot \pi(\gamma)(\lambda_1 \lambda_2) &= \bar{\rho}(\gamma)(n \lambda_1 \lambda_2) = \\ \bar{\rho}(\gamma)(n \lambda_1) \cdot \pi(\gamma)(\lambda_2) &= \bar{\rho}(\gamma)(n) \cdot \pi(\gamma)(\lambda_1) \cdot \pi(\gamma)(\lambda_2), \end{aligned}$$

for any $n \in N$. Comparing first and last terms shows $\pi(\gamma)$ is a homomorphism.

To complete the proof, it suffices to show the map

$$\pi : \Gamma \rightarrow \text{Hom}(\Lambda)$$

is a homomorphism. Let $\gamma_1, \gamma_2 \in \Gamma$, $\lambda \in \Lambda$, and compute that

$$\begin{aligned} \bar{\rho}(\gamma_1 \gamma_2)(n) \cdot \pi(\gamma_1 \gamma_2)(\lambda) &= \bar{\rho}(\gamma_1 \gamma_2)(n \lambda) = \\ \bar{\rho}(\gamma_1)(\bar{\rho}(\gamma_2)(n \lambda)) &= \bar{\rho}(\gamma_1)[\bar{\rho}(\gamma_2)(n) \cdot \pi(\gamma_2)(\lambda)] = \\ [\bar{\rho}(\gamma_1)(\bar{\rho}(\gamma_2)n)] [\pi(\gamma_1)(\pi(\gamma_2)(\lambda))] &= \bar{\rho}(\gamma_1 \gamma_2)(n) \cdot \pi(\gamma_1)(\pi(\gamma_2)(\lambda)), \end{aligned}$$

for each $n \in N$. Comparing first and last terms shows π is a homomorphism. \square

Remark IV.4. In view of Remark II.22, we may consider π as taking values in the discrete subgroup $\text{Aut}(N/\Lambda) \subset \text{Aut}(N)$.

Remark IV.5. In case N is abelian, the nilmanifold is a torus \mathbb{T}^n . By Lemma IV.3, there is a homomorphism

$$\pi : \Gamma \rightarrow \text{Aut}(\mathbb{Z}^n) = GL(n, \mathbb{Z}).$$

Using covering space theory, it is possible to show that π is nothing more than the homomorphism

$$\rho_* : \Gamma \rightarrow \text{Aut}(H_1(\mathbb{T}^n))$$

coming from the induced action on first homology.

Lemma IV.6. *There is a map*

$$A : \Gamma \times N/\Lambda \rightarrow N$$

satisfying

$$\bar{\rho}(\gamma)(n) = \pi(\gamma)(A(\gamma, [n])n)$$

for each $n \in N$ and $\gamma \in \Gamma$. For a fixed $\gamma \in \Gamma$, $A(\gamma, \cdot) \in C^k(N/\Lambda, N)$.

Proof. Define

$$A : \Gamma \times N \rightarrow N$$

by

$$\bar{\rho}(\gamma)(n) = \pi(\gamma)(A(\gamma, n)n).$$

We need to check that for a fixed $\gamma \in \Gamma$, $A(\gamma, \cdot)$ descends to a N valued function on N/Λ . Let $n \in N$ and $\lambda \in \Lambda$. We compute that

$$\begin{aligned} \pi(\gamma)(A(\gamma, n\lambda)n\lambda) &= \bar{\rho}(\gamma)(n\lambda) = \\ \bar{\rho}(\gamma)(n) \cdot \pi(\gamma)(\lambda) &= \pi(\gamma)(A(\gamma, n)n) \cdot \pi(\gamma)(\lambda) = \\ \pi(\gamma)(A(\gamma, n)n\lambda), \end{aligned}$$

which upon comparing the first and last term yields $A(\gamma, n\lambda) = A(\gamma, n)$ as required. \square

Lemma IV.7. *For each $\gamma_1, \gamma_2 \in \Gamma$ and $[n] \in N/\Lambda$,*

$$A(\gamma_1\gamma_2, [n]) = \pi(\gamma_2^{-1})[A(\gamma_1, \rho(\gamma_2)[n])] \cdot A(\gamma_2, [n]).$$

Proof. We simply compute that

$$\begin{aligned} \pi(\gamma_1\gamma_2)[A(\gamma_1\gamma_2, [n])n] &= \bar{\rho}(\gamma_1\gamma_2)(n) = \bar{\rho}(\gamma_1)[\bar{\rho}(\gamma_2)(n)] = \\ \bar{\rho}(\gamma_1)[\pi(\gamma_2)(A(\gamma_2, [n])n)] &= \pi(\gamma_1)\{A(\gamma_1, \rho(\gamma_2)[n]) \cdot \pi(\gamma_2)(A(\gamma_2, [n])n)\} = \\ \pi(\gamma_1\gamma_2)[\pi(\gamma_2^{-1})(A(\gamma_1, \rho(\gamma_2)[n])) \cdot A(\gamma_2, [n])n], \end{aligned}$$

which upon comparing first and last terms finishes the proof. \square

Lemma IV.8. *Define the maps*

$$C : \Gamma \rightarrow C^k(N/\Lambda, N) \quad L : \Gamma \rightarrow \text{Hom}(C^k(N/\Lambda, N))$$

by

$$C(\gamma) := A(\gamma^{-1}, \cdot) \quad (L(\gamma) \cdot f)(\cdot) := \pi(\gamma)(f(\rho(\gamma^{-1})(\cdot)))$$

for $\gamma \in \Gamma$ and $f \in C^k(N/\Lambda, N)$. Then for each $\gamma_1, \gamma_2 \in \Gamma$,

$$C(\gamma_1\gamma_2) = L(\gamma_1)(C(\gamma_2)) \cdot C(\gamma_1).$$

Proof. This follows directly from replacing γ_1 with γ_2^{-1} and replacing γ_2 with γ_1^{-1} in the relation from Lemma IV.7. \square

4.3 Weakly Hyperbolic Actions of Property (T) Groups on Tori

In this section we consider volume preserving weakly hyperbolic actions of discrete property (T) groups Γ on tori. We argue that whenever such an action is covered by a Γ action on \mathbb{R}^n , the representation coming from the homomorphism $\Gamma \rightarrow \text{Out}(\pi_1(\mathbb{T}^n))$ cannot split as a nontrivial direct sum of subrepresentations, one of which is isometric.

Throughout, let Γ denote a discrete Kazhdan group and let

$$\rho : \Gamma \rightarrow \text{Diff}^2(\mathbb{T}^n)$$

be a volume preserving weakly hyperbolic action covered by an action,

$$\bar{\rho} : \Gamma \rightarrow \text{Diff}^2(\mathbb{R}^n).$$

The action ρ induces a homomorphism $\Gamma \rightarrow \text{Out}(\pi_1(\mathbb{T}^n))$ that lifts to the homomorphism π from Lemma IV.3. Switching to additive notation, π satisfies

$$\bar{\rho}(\gamma)(x + z) = \bar{\rho}(\gamma)(x) + \pi(\gamma)(z),$$

for each $\gamma \in \Gamma$, $x \in \mathbb{R}^n$, and $z \in \mathbb{Z}^n$. For simplicity, we also let π denote the representation $\Gamma \rightarrow \text{GL}(n, \mathbb{R}^n)$ induced by the homomorphism π . Finally, let ρ_0 denote the linear action on the torus induced by π . We assume that there is a direct sum decomposition of \mathbb{R}^n into Γ invariant subspaces I and H so that the restriction of π to I , π^I , acts isometrically in I and will argue that $I = \{0\}$.

The main idea in the argument below is that the representation π coarsely approximates the lifted action $\bar{\rho}$. More precisely, by covering space theory, the two actions agree when restricted to $\mathbb{Z}^n \subset \mathbb{R}^n$. Weak hyperbolicity requires that all tangent directions are spanned by directions uniformly contracted by some element of the group under $\bar{\rho}$. The same should also be true for the discretely approximating action π . Therefore, π cannot have a nontrivial isometric invariant subspace I .

Making this line of reasoning precise involves analyzing a cocycle (first introduced in [MQ01]) which measures the difference between the lifted action and the induced action on the fundamental group. The heart of the argument lies in the following:

Proposition IV.9. *Assume the hypotheses above. Then there is a continuous map*

$$\phi : \mathbb{R}^n \rightarrow I$$

of the form $\phi(x) = \text{proj}_I(x) + \sigma([x])$ where $\sigma \in C^0(\mathbb{T}^n, I)$ such that $\pi^I(\gamma)(\phi(x)) = \phi(\bar{\rho}(\gamma)x)$.

Proof. By Lemma IV.6, there is a map

$$A : \Gamma \times \mathbb{T}^n \rightarrow \mathbb{R}^n$$

satisfying

$$\bar{\rho}(\gamma)x = \pi(\gamma)(x + A(\gamma, [x])).$$

Following A by projection to the subspace I parallel to the complementary subspace H , we obtain a map

$$A^I : \Gamma \times \mathbb{T}^n \rightarrow I.$$

Since H is an invariant complement to I , A^I solves the equation

$$\text{proj}_I(\bar{\rho}(\gamma)x) = \pi^I(\gamma)(\text{proj}_I(x) + A^I(\gamma, [x])).$$

Next consider the map L given by Lemma IV.8. As ρ preserves volume and π^I acts isometrically, the map $\text{proj}_I \circ L$ extends to a unitary representation

$$L^I : \Gamma \rightarrow U(L^2(\mathbb{T}^n, I))$$

defined by

$$(L(\gamma)f)([x]) := \pi^I(\gamma)f(\rho(\gamma^{-1})[x]).$$

Next, we consider the map C given by Lemma IV.8. Following C by projection to the subspace I yields a map

$$C^I : \Gamma \rightarrow C^k(\mathbb{T}^n, I) \subset L^2(\mathbb{T}^n, I).$$

The identity of Lemma IV.8 written in additive notation implies that C^I is a 1-cocycle (II.26) in $Z^1(\Gamma, L^I)$.

By Theorem II.32 there is a $\sigma \in L^2(\mathbb{T}^n, I)$ such that

$$(**) \quad A^I(\gamma^{-1}) = \sigma - L^I(\gamma)\sigma,$$

holds as an equation in $L^2(\mathbb{T}^n, I)$ for every $\gamma \in \Gamma$. Next, we argue that σ agrees almost everywhere with a continuous function, and consequently, that $(**)$ holds as an equation in $C^0(\mathbb{T}^n, I)$. In view of Corollary III.11, it suffices to show that for a partially hyperbolic diffeomorphism $\rho(\gamma) \in PH_\mu^2(\mathbb{T}^n)$, there is a uniform constant $C > 0$ such that the restriction of σ to almost all leaves of the stable foliation $\mathcal{W}_{\rho(\gamma)}^s$ is almost everywhere C -Lipschitz.

For this step, it helps to first simplify notation by writing γ instead of $\rho(\gamma)$ and $\hat{\gamma}$ instead of $\pi^I(\gamma^{-1})$. Furthermore we will let f denote the C^2 function $A^I(\gamma)$. In this simplified notation, $(**)$ implies that there is a full volume γ -invariant set $V \subset \mathbb{T}^n$ so that

$$(***) \quad \sigma(x) = f(x) + \hat{\gamma}\sigma(\gamma x)$$

for all $x \in V$.

By the ergodic decomposition (II.3), there is a full volume γ -invariant set $M' \subset \mathbb{T}^n$, a standard Borel probability space (Ω, ν) , a γ -invariant Borel map

$$\psi : M' \rightarrow \Omega,$$

and a Borel map

$$\xi : \Omega \rightarrow \text{Prob}(M')$$

such that

$$\mu = \int_{\Omega} \xi(\omega) d\nu,$$

where each $\xi(\omega)$ is a quasi-invariant ergodic probability measure. As in the proof of Lemma III.12, for each $g \in L^1$ define the full volume set $B_g \subset M'$ by

$$B_g := \{x \in M' \mid g^+(x) = \int_{M'} g d\xi(\psi(x))\}$$

and the full volume set

$$B_0 = \bigcap_{g \in C^0(M)} B_g.$$

By Lusin's theorem there is a sequence of compact sets $K_j \subset K_{j+1}$ such that the restriction of σ to K_j is uniformly continuous and $\mu(K_j) > 1 - \frac{1}{2^j}$. Let $K = \bigcup_j K_j$ and define

$$B_K := \{x \in M' \mid \xi(\psi(x))(K) = 1\}.$$

It is straightforward to argue that B_K is a full volume Borel set. Finally, let

$$G := V \cap B_K \cap_j B_{\chi(K_j)} \cap B_0.$$

Then G is a full volume Borel set and by absolute continuity (II.11), there is a full volume subset of points $E \subset G$ so that G is conull in $\mathcal{W}_\gamma^s(x)$ whenever $x \in E$.

Fix $x \in E$ and y_1, y_2 in $G \cap \mathcal{W}_\gamma^s(x)$. Since $y_1, y_2 \in B_0$, the Hopf argument shows that $\xi(\psi(y_1)) = \xi(\psi(y_2))$. Let m denote this ergodic measure. Since $y_1 \in B_K$, $m(K) = 1$ so that there is a large enough j for which $m(K_j) > \frac{1}{2}$. Since $y_1, y_2 \in B_{\chi(K_j)}$ there are infinitely many n for which $\gamma^n y_1$ and $\gamma^n y_2$ both lie in K_j . Let $\epsilon > 0$. As γ contracts distances in $\mathcal{W}_\gamma^s(x)$ by some constant $\lambda < 1$ and by uniform continuity of σ in K_j , there is a large enough N so that $d_C(\sigma(\gamma^N y_1), \sigma(\gamma^N y_2)) < \epsilon$. By iterating (**), we therefore obtain that

$$\begin{aligned} d_I(\sigma(y_1), \sigma(y_2)) &= \left\| \sum_{i=0}^{N-1} \widehat{\gamma}^i (f(\gamma^i y_1) - f(\gamma^i y_2)) + (\sigma(\gamma^N y_1) - \sigma(\gamma^N y_2)) \right\| \\ &\leq \sum_{i=0}^{N-1} L d_M(\gamma^i y_1, \gamma^i y_2) + \epsilon \leq \sum_{i=0}^{\infty} L \lambda^i d_M(y_1, y_2) + \epsilon, \end{aligned}$$

where L is a Lipschitz constant for f and λ is the contraction constant for γ . This concludes the proof that σ is continuous.

To finish the proof, it remains to show that the map ϕ as defined in the statement of the proposition is equivariant. We compute that

$$\phi(\bar{\rho}(\gamma)x) = \text{proj}_I(\bar{\rho}(\gamma)x) + \sigma(\rho(\gamma)[x]) =$$

$$\begin{aligned}
& \text{proj}_I(\pi(\gamma)x + \pi(\gamma)A(\gamma, [x])) + \sigma(\rho(\gamma)[x]) = \\
& \pi^I(\gamma) \text{proj}_I(x) + \pi^I(\gamma)A^I(\gamma, [x]) + \sigma(\rho(\gamma)[x]) = \\
& \pi^I(\gamma) \text{proj}_I(x) + \pi^I(\gamma)\sigma([x]) = \pi^I(\gamma)\phi(x),
\end{aligned}$$

where the penultimate equality follows from (**).

□

Lemma IV.10. *With the same hypotheses as in Proposition IV.9, the map ϕ is constant.*

Proof. Fix a lift $\widetilde{\mathcal{W}}^-(x)$ of a stable leaf of a partially hyperbolic diffeomorphism $\rho(\gamma)$ and let $y \in \widetilde{\mathcal{W}}^-(x)$. We argue by contradiction and suppose that $d_0 := d_I(\phi(x), \phi(y)) > 0$.

By uniform continuity of ϕ there is a $\delta > 0$ so that $d(x, y) < \delta$ implies $d_I(\sigma(x), \sigma(y)) < \frac{d_0}{2}$. For sufficiently large n , $d(\bar{\rho}(\gamma^n)x, \bar{\rho}(\gamma^n)y) < \delta$. Therefore,

$$d_0 = d_I(\phi(x), \phi(y)) = d_I(\pi^I(\gamma^n)\phi(x), \pi^I(\gamma^n)\phi(y)) = d_I(\phi(\bar{\rho}(\gamma^n)x), \phi(\bar{\rho}(\gamma^n)y)) < \frac{d_0}{2}.$$

The previous line yields a contradiction. Therefore, ϕ must be constant on lifts of stable leaves. By Proposition III.5, ϕ is locally constant and therefore constant by the connectedness of \mathbb{T}^n . □

Theorem IV.11. *With the same hypotheses as in Proposition IV.9, the subspace I is trivial.*

Proof. We argue by contradiction. Suppose I has positive dimension. Since ϕ is defined by adding a bounded map to the projection map to the subspace I , ϕ is unbounded. However, this contradicts Lemma IV.10. □

It is instructive to see how the arguments in this section rely on the abelian group structure on \mathbb{R}^n , while the supporting lemmas from the previous section work for nilpotent group structures. Essentially, the difference rests on the fact that $L^2(\mathbb{T}^n, \mathbb{R}^n)$ has a vector space structure given by pointwise addition while no such structure exists in the non-abelian case. Nevertheless, the next and final theorem of this section shows that Theorem IV.11 generalizes in the non-abelian nilpotent case to the complement of the first commutator subalgebra of the Lie algebra of N . Unfortunately, its formulation is a little long:

Theorem IV.12. *Suppose that*

$$\bar{\rho} : \Gamma \rightarrow \text{Diff}^2(N)$$

is an action of a discrete property (T) group Γ on a connected and simply connected nilpotent Lie group N which covers an action

$$\rho : \Gamma \rightarrow \text{Diff}^k(N/\Lambda)$$

on a nilmanifold. Let \mathfrak{n} denote the Lie algebra of N and

$$d\pi : \Gamma \rightarrow \text{Aut}(\mathfrak{n}) \subset GL(\mathfrak{n})$$

denote the representation defined by $d\pi(\gamma) := d(\pi(\gamma))_e$, where $\gamma \in \Gamma$, π is given by Lemma IV.3, and $e \in N$ denotes the identity element. Furthermore, suppose that there is a $d\pi(\Gamma)$ invariant complement V to the subrepresentation on $[\mathfrak{n}, \mathfrak{n}]$. Then the representation $d\pi|_V$ given by restriction to V does not split nontrivially into subrepresentations, one of which is isometric.

Proof. Let $\exp : \mathfrak{n} \rightarrow N$ be the exponential map. It is well known that \exp is a diffeomorphism. Denote its inverse by $\log : N \rightarrow \mathfrak{n}$. Let C and L be the maps

defined in Lemma IV.8. Define the maps

$$c : \Gamma \rightarrow C^k(N/\Lambda, \mathfrak{n}) \quad l : \Gamma \rightarrow \text{Hom}(C^k(N/\Lambda, \mathfrak{n}))$$

by

$$c(\gamma) := \log(C(\gamma)) \quad (l(\gamma) \cdot f)(\cdot) := \log(L(\gamma) \cdot \exp(f(\cdot)))$$

for $f \in C^k(N/\Lambda, \mathfrak{n})$.

By naturality, π and $d\pi$ are exp equivariant. It follows that $(l(\gamma) \cdot f)(\cdot) = d\pi(\gamma)f(\rho(\gamma^{-1})(\cdot))$. Next we calculate that

$$\begin{aligned} c(\gamma_1\gamma_2)(\cdot) &= \log[C(\gamma_1\gamma_2)](\cdot) = \log[(L(\gamma_1)C(\gamma_2) \cdot C(\gamma_1))](\cdot) = \\ &= \log[\pi(\gamma_1)\{\exp(c(\gamma_2)(\rho(\gamma_1^{-1}(\cdot))))\} \cdot \exp(c(\gamma_1))](\cdot) = \\ &= \log\{\exp[d\pi(\gamma_1)(c(\gamma_2)(\rho(\gamma_1^{-1}(\cdot))))] \exp[c(\gamma_1)]\}(\cdot), \end{aligned}$$

holds as an equation in $C^k(N/\Lambda, \mathfrak{n})$. Following c by projection to the subspace V , defines a map

$$c^V : \Gamma \rightarrow C^k(N/\Lambda, V).$$

Similarly, projection to the subspace V induces a representation

$$l^V : \Gamma \rightarrow \text{Hom}(C^k(N/\Lambda, V)).$$

It follows from the Baker-Campbell-Hausdorff formula (II.16) and the above equation that c^V is a one cocycle for the representation l^V . The proof of Theorem IV.11 now completes this proof. \square

Remark IV.13. It seems likely that Theorem IV.11 fully generalizes to nilmanifolds.

4.4 Application to Zimmer Program

This section is devoted to the problem of classifying volume preserving weakly hyperbolic higher rank lattice actions on tori. Throughout we assume that the acting group Γ is a higher rank lattice. We'll deduce that all weakly hyperbolic C^2 volume preserving actions on a torus that lift to the universal cover are semiconjugate to the linear action coming from the fundamental group when restricted to a finite index subgroup of the acting lattice. We also argue that this semiconjugacy is a conjugacy under the additional hypothesis that the leaves of the lift of the unstable foliation of a partially hyperbolic group element are *quasi-isometrically embedded* (Definition IV.20 below) in \mathbb{R}^n .

Let

$$\rho : \Gamma \rightarrow \text{Diff}^2(\mathbb{T}^n)$$

be a volume preserving action covered by an action $\bar{\rho}$ on \mathbb{R}^n . Let π be the induced homomorphism from Lemma IV.3 and ρ_0 be the affine action induced by π . In [MQ01], they make the following:

Definition IV.14. A representation $\pi : \Gamma \rightarrow \text{GL}(n, \mathbb{R})$ is said to be *weakly hyperbolic* if the Zariski closure of $\pi(\Gamma) \subset \text{GL}(n, \mathbb{R}^n)$ is not precompact in any of its nontrivial subrepresentations.

The next theorem is from [MQ01]. We remark that although they assume the existence of a periodic point for the action, their proof works equally well under the weaker assumption that the action lifts to \mathbb{R}^n .

Theorem IV.15 (Theorem 6.10 in [MQ01]). *Let Γ , ρ , and $\bar{\rho}$ be as above. If π_ρ is weakly hyperbolic, then there exists a finite index subgroup $\Gamma' < \Gamma$ and a map $\phi \in C^0(\mathbb{T}^n)$, unique in the homotopy class of the identity such that $\phi \circ \rho(\gamma) = \rho_0(\gamma) \circ \phi$*

for all $\gamma \in \Gamma'$.

Next we establish a complimentary statement.

Theorem IV.16. *Let Γ , ρ , and $\bar{\rho}$ be as above. If ρ is weakly hyperbolic, then π_ρ is weakly hyperbolic.*

Proof. Suppose that the Zariski closure of $\pi(\Gamma)$ is precompact in $\mathrm{GL}(I)$ for some invariant subspace I . By Corollary II.25, there is an invariant subspace complementary to I . Since Γ has property (T), Theorem IV.11 implies $I = 0$. \square \square

Combining Theorems IV.15 and IV.16 yields the next corollary, a first step at showing all weakly hyperbolic higher rank lattice actions on tori are affine algebraic.

Corollary IV.17. *Let Γ , ρ , and $\bar{\rho}$ be as above. If ρ is weakly hyperbolic, then after passing to a finite index subgroup of $\Gamma' \subset \Gamma$, ρ is C^0 -semiconjugate to the affine action ρ_0 coming from the homomorphism $\Gamma \rightarrow \mathrm{Out}(\pi_1(\mathbb{T}^n))$ by a map ϕ which is unique in the homotopy class of the identity.*

Corollary IV.18. *Let Γ and ρ be as above and let π be the homomorphism of Lemma IV.3. If either ρ has a fixed point or $H^2(\Gamma, \pi) = 0$, then after passing to a finite index subgroup, ρ is C^0 -semiconjugate to the affine action ρ_0 coming from the homomorphism $\Gamma \rightarrow \mathrm{Out}(\pi_1(\mathbb{T}^n))$ by a map ϕ which is unique in the homotopy class of the identity.*

Proof. By Corollary IV.17, it suffices to show that ρ is covered by an action $\bar{\rho}$ on \mathbb{R}^n . If $H^2(\Gamma, \pi) = 0$, Theorem II.28 and Remark II.29 implies that ρ is covered by an action $\bar{\rho}$. If ρ has a fixed point $p \in \mathbb{T}^n$, choose a point $\tilde{p} \in \mathbb{R}^n$ in the fiber above p . For each $\gamma \in \Gamma$, lift $\rho(\gamma)$ to the diffeomorphism $\bar{\rho}(\gamma)$ of \mathbb{R}^n fixing \tilde{p} . This defines an action of the free group on Γ , $F(\Gamma)$, on \mathbb{R}^n with fixed point \tilde{p} . Let K be the kernel

of the natural homomorphism $F(\Gamma) \rightarrow \Gamma$. For $k \in K$, $\bar{\rho}(k)$ covers the identity map of \mathbb{T}^n so that for each $x \in \mathbb{R}^n$ there is a $z(n, k) \in \mathbb{Z}^n$ satisfying

$$\bar{\rho}(k)(n) = n + z(n, k.)$$

As \mathbb{R}^n is connected and \mathbb{Z}^n is discrete, $z(\cdot, k)$ is constant on \mathbb{R}^n . Since $\bar{\rho}(k)(\tilde{p}) = \tilde{p}$, $z(\tilde{p}, k) = 0$, whence $\bar{\rho}(k)$ acts as the identity on \mathbb{R}^n . It follows that $\bar{\rho}$ defines an action of Γ covering ρ . \square

Remark IV.19. By II.30 the cohomological vanishing condition in Corollary IV.18 is satisfied for uniform lattices in simple Lie groups of rank three and higher.

Note that Corollary IV.17 does not complete the continuous classification of volume preserving weakly hyperbolic actions of higher rank lattices on tori. To complete the classification, one needs to show that the semiconjugacy ϕ of is actually a homeomorphism. Since ϕ is homotopic to the identity map, it must be surjective. Next we argue that ϕ is a C^0 conjugacy provided that the lifts of unstable leaves to \mathbb{R}^n have intrinsic distances comparable to Euclidean distance for some partially hyperbolic group element. Following [Bri03] we make the following:

Definition IV.20. A foliation \mathcal{W} of a simply connected metric space (X, d) is said to be *quasi-isometric* if there are uniform constants $a, b > 0$ such that for each $x \in X$ and $y \in \mathcal{W}(x)$, $d_{\mathcal{W}(x)}(x, y) \leq ad_X(x, y) + b$.

Proposition IV.21. *Let Γ' , ρ , $\bar{\rho}$, π , and ρ_0 be as in Corollary IV.17. Let ϕ be the unique continuous map homotopic to the identity such that $\phi \circ \rho(\gamma) = \rho_0(\gamma) \circ \phi$ for all $\gamma \in \Gamma'$. If there exists a group element $\gamma \in \Gamma'$ such that $\rho(\gamma)$ is partially hyperbolic and such that the lift $\widetilde{\mathcal{W}}_{\rho(\gamma)}^u$ of the unstable foliation $\mathcal{W}_{\rho(\gamma)}^u$ to \mathbb{R}^n is quasi-isometric, then ϕ is a homeomorphism.*

Proof. The map ϕ has degree one and is therefore surjective. To prove injectivity, it suffices to prove that ϕ is locally injective. It therefore suffices to show that a lift of ϕ to \mathbb{R}^n is locally injective. First we argue that ϕ may be equivariantly lifted with respect to $\bar{\rho}$ and π . Choose a lift $\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of ϕ and define

$$\theta : \Gamma' \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

by $\theta(\gamma, x) = \tau(\bar{\rho}(\gamma)(x)) - \pi(\gamma)(\tau(x))$. Since $\tau(\bar{\rho}(\gamma)(x))$ and $\pi(\gamma)(\tau(x))$ project to the same point in the torus, $\theta(\Gamma' \times \mathbb{R}^n) \subset \mathbb{Z}^n$. As \mathbb{Z}^n is discrete, each $\theta(\gamma, \cdot)$ is constant so that we may alternatively view θ as a map

$$\theta : \Gamma' \rightarrow \mathbb{Z}^n \subset \mathbb{R}^n.$$

Viewed this way, θ is a one cocycle over π . Indeed,

$$\begin{aligned} \theta(\gamma_1\gamma_2) &= \tau(\bar{\rho}(\gamma_1\gamma_2)(0)) - \pi(\gamma_1\gamma_2)(\tau(0)) = \\ &= \tau(\bar{\rho}(\gamma_1)(\bar{\rho}(\gamma_2)(0))) - \pi(\gamma_1)(\pi(\gamma_2)(\tau(0))) = \\ &= \theta(\gamma_1, \bar{\rho}(\gamma_2)(0)) + \pi(\gamma_1)(\tau(\bar{\rho}(\gamma_2)(0))) - \pi(\gamma_1)(\pi(\gamma_2)(\tau(0))) = \\ &= \theta(\gamma_1) + \pi(\gamma_1)(\theta(\gamma_2)), \end{aligned}$$

for each $\gamma_1, \gamma_2 \in \Gamma'$. Since $H^1(\Gamma', \pi) = 0$ (II.27), there is a $v \in \mathbb{R}^n$ so that

$$\theta(\gamma) = \pi(\gamma)(v) - v,$$

for each $\gamma \in \Gamma'$. Define

$$\bar{\phi} : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

by $\bar{\phi}(x) = \tau(x) + v$. It is straightforward to check that $\bar{\phi}$ is equivariant with respect to $\bar{\rho}$ and π . It therefore remains to show $\bar{\phi}$ is a cover of ϕ . First note that $\bar{\phi}$ descends to a map ϕ' homotopic to the identity on \mathbb{T}^n since

$$\bar{\phi}(x + z) = \tau(x + z) + v = \tau(x) + z + v = \bar{\phi}(x) + z,$$

for each $x \in \mathbb{R}^n$ and $z \in \mathbb{Z}^n$. Moreover, ϕ' is equivariant with respect to ρ and ρ_0 and therefore coincides with ϕ by uniqueness.

To finish the argument, we show that $\bar{\phi}$ is locally injective. Since ϕ is homotopic to the identity there is some $M > 0$ so that $\|\bar{\phi}(x) - x\| < M$. If $\bar{\phi}$ is not locally injective, we may choose $x, y \in \mathbb{R}^n$ with the same image and sufficiently close so that there exists a piecewise C^1 curve $\sigma : [0, 1] \rightarrow \mathbb{R}^n$ satisfying $\sigma(0) = x$, $\sigma(1) \in \widetilde{\mathcal{W}}_{\rho(\gamma)}^u(y)$, and $\dot{\sigma} \in E_{\rho(\gamma)}^s \oplus E_{\rho(\gamma)}^c$. By equivariance, $\bar{\phi}(\bar{\rho}(\gamma^n)(x)) = \bar{\phi}(\bar{\rho}(\gamma^n)(y))$ for each natural number n . Therefore,

$$\begin{aligned} 2M &> \|\bar{\rho}(\gamma^n)x - \bar{\rho}(\gamma^n)y\| \geq \|\bar{\rho}(\gamma^n)y - \bar{\rho}(\gamma^n)\sigma(1)\| - \|\bar{\rho}(\gamma^n)\sigma(1) - \bar{\rho}(\gamma^n)x\| \\ &\geq \frac{1}{a}d_{\widetilde{\mathcal{W}}_\gamma^u}(\bar{\rho}(\gamma^n)\sigma(1), \bar{\rho}(\gamma^n)y) - K \text{length}(\bar{\rho}(\gamma^n)\sigma) - \frac{b}{a}, \end{aligned}$$

a contradiction since the last term of this inequality grows unbounded with n .

□

Next we show that if the semiconjugacy ϕ were more regular, then it must also be injective. We'll need the following theorem during the course of the argument.

Theorem IV.22 ([WM94b], Witte-Morris). *Let Γ be a lattice in a connected semisimple Lie group G with no compact factors and let N be a connected, simply connected nilpotent group and $\Lambda \subset N$ a lattice. Suppose G acts on N by automorphisms such that Γ normalizes Λ and so acts on N/Λ . Then any ergodic Γ invariant probability measure μ on N/Λ is homogeneous for a subgroup of $\Gamma \subset N$.*

Proposition IV.23. *Let Γ' , ρ , $\bar{\rho}$, π , and ρ_0 be as in Corollary IV.17. Let ϕ be the unique continuous map homotopic to the identity such that $\phi \circ \rho(\gamma) = \rho_0(\gamma) \circ \phi$ for all $\gamma \in \Gamma'$. If ϕ is once continuously differentiable, then ϕ is a homeomorphism.*

Proof. It suffices to prove ϕ is locally injective. It follows from the chain rule for differentiation, equivariance of ϕ , and the fact that ρ and ρ_0 are volume preserving

actions that the Jacobian $J(\phi)$ of ϕ is a ρ invariant function. By (III.14) ρ is ergodic with respect to volume so that $J(\phi)$ is almost everywhere constant. By continuity J is everywhere constant. To show this constant is nonzero, note that $\phi_*(\text{vol})$ is a full support ρ_0 invariant measure. By Theorem IV.22, $\phi_*(\text{vol}) = \text{vol}$, whence $J > 0$. \square

The next proposition shows that a certain cohomology group vanishing implies that there is a more regular semiconjugacy ϕ .

Proposition IV.24. *Let Γ be a group, ρ a C^1 action of Γ on \mathbb{T}^n covered by an action $\bar{\rho}$ on \mathbb{R}^n , π the induced homomorphism, and ρ_0 the induced affine action. Suppose $H^1(\Gamma, \lambda) = 0$, where*

$$\lambda : \Gamma \rightarrow \text{Hom}(C^2(\mathbb{T}^n, \mathbb{R}^n))$$

is defined by $(\lambda(\gamma) \cdot f)(\cdot) := \pi(\gamma)(f(\rho(\gamma^{-1})(\cdot)))$ for $f \in C^1(\mathbb{T}^n, \mathbb{R}^n)$. Then there is a map $\phi \in C^1(\mathbb{T}^n)$ homotopic to the identity such that $\rho_0(\gamma) \circ \phi = \phi \circ \rho(\gamma)$ for each $\gamma \in \Gamma$.

Proof. In the same notation as in Lemma IV.8, C is a 1-cocycle for the representation λ . By assumption, $H^1(\Gamma, \lambda) = 0$ so that there is a $\sigma \in C^1(\mathbb{T}^n, \mathbb{R}^n)$ satisfying

$$C(\gamma) = \sigma - \lambda(\gamma) \cdot \sigma$$

for each $\gamma \in \Gamma$. Define

$$\bar{\phi} : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

by $\bar{\phi}(x) = x + \sigma([x])$ for $x \in \mathbb{R}^n$. As $\bar{\phi}(x+z) = \bar{\phi}(x)$ for each $z \in \mathbb{Z}^n$, $\bar{\phi}$ descends to a C^1 map of the torus homotopic to the identity.

Next we check that ϕ is an equivariant map. It suffices to show that $\bar{\phi}$ is equivariant. We compute that

$$\bar{\phi}(\bar{\rho}(\gamma)(x)) = \bar{\rho}(\gamma)(x) + \sigma(\rho(\gamma)[x]) =$$

$$\begin{aligned} \bar{\rho}(\gamma)(x) + \pi(\gamma)[\sigma([x]) - C(\gamma^{-1})([x])] = \\ \pi(\gamma)(x + A(\gamma, [x])) + \pi(\gamma)(\sigma([x]) - A(\gamma, [x])) = \pi(\gamma)(\bar{\phi}(x)), \end{aligned}$$

as required. □

Remark IV.25. The semiconjugacy ϕ is unique provided all such lift equivariantly to \mathbb{R}^n . This follows since if τ were another with an equivariant lift $\bar{\tau}$ then σ' defined by $\sigma'([x]) = \bar{\tau}(x) - x$ solves $C(\gamma) = \sigma - \lambda(\gamma) \cdot \sigma$ for each $\gamma \in \Gamma$.

CHAPTER V

Conclusion

In this concluding chapter, we pose some questions related to the present work.

The first question we address is well known and comes from the theory of stable ergodicity for partially hyperbolic diffeomorphisms. We first make the following definitions:

Definition V.1. Let $f \in \text{Diff}^2(M)$ be a volume preserving partially hyperbolic diffeomorphism. The diffeomorphism f is said to have the *accessibility property* if any point in M can be reached from any other along an *su-path*, i.e. a concatenation of finitely many subpaths each of which lies entirely in a single leaf of \mathcal{W}_f^s or of \mathcal{W}_f^u .

Given a point $p \in M$, its *accessibility class* is the set $q \in M$ reachable from p by an *us-path*.

Definition V.2. A volume preserving partially hyperbolic diffeomorphism $f \in \text{Diff}^2(M)$ is *essentially accessible* if every measurable set that is a union of entire accessibility classes is null or conull.

The following conjecture has been made by Pugh and Shub.

Conjecture V.3. *Let f be a C^2 volume preserving partially hyperbolic diffeomorphism. Then essential accessibility implies ergodicity.*

Perhaps the approach developed in this thesis for proving the ergodicity of weakly hyperbolic group actions could be used as an approach for Conjecture V.3. More specifically, we seek a positive answer to the following question:

Question V.4. *Let \mathcal{F}_1 and \mathcal{F}_2 be absolutely continuous foliations with the essential accessibility property. Suppose the restriction of $f \in L^2(M)$ to almost all leaves of \mathcal{F}_1 and \mathcal{F}_2 is almost everywhere K -Lipschitz. Then does it follow that f agrees almost everywhere with a continuous function?*

In view of the Frank and Manning classification of Anosov diffeomorphisms on tori (IV.1 and IV.2), one might suspect that a similar classifications should hold for partially hyperbolic diffeomorphisms on tori. Currently, this question appears to be out of reach. A necessary condition for a partially hyperbolic diffeomorphism on a torus to be continuously conjugated to its induced action on first homology is for its stable foliation \mathcal{W}_f^s to lift quasi-isometrically to \mathbb{R}^n . Additionally, we seek a positive answer to the following question in view of Proposition IV.21.

Question V.5. *Let $f \in \text{Diff}^2(\mathbb{T}^n)$ be a volume preserving partially hyperbolic diffeomorphism. Does the stable foliation $\mathcal{W}_f^s(\mathbb{T}^n)$ lift quasi-isometrically to \mathbb{R}^n ?*

The results in Chapter 4 suggest that it should be possible to give a continuous classification for volume preserving weakly hyperbolic higher rank lattice actions on tori. For completeness, we record the following:

Conjecture V.6. *Let Γ be a higher rank lattice and ρ a volume preserving weakly hyperbolic action of Γ on a nilmanifold N/Λ . Then ρ is continuously conjugated to an affine algebraic action ρ_0 on N/Λ .*

There are several obstacles to applying the approach used in Chapter 4. First the results in Chapter 4 apply to C^2 actions as opposed to C^1 actions. Specifically, we

required that the stable foliations of group elements that act by partially hyperbolic diffeomorphisms be absolutely continuous. Secondly, our approach requires that the action ρ be covered by an action $\bar{\rho}$ on the connected, simply connected nilpotent Lie group N . We pose this as a separate question.

Question V.7. *Let ρ be an action of a higher rank lattice Γ on a nilmanifold N/Λ . Must the action ρ be covered by an action $\bar{\rho}$ on N ?*

A third obstacle to using the techniques of Chapter 4 is that Corollary IV.17 only gives a semiconjugacy between a weakly hyperbolic action ρ and a weakly hyperbolic affine algebraic action ρ_0 . While we show that such a semiconjugacy is a homeomorphism under various additional hypotheses, we are unable to answer the following general:

Question V.8. *Let ρ be a weakly hyperbolic action of a higher rank lattice on a nilmanifold. Suppose that ρ is semiconjugate to a weakly hyperbolic affine algebraic action ρ_0 via a semiconjugacy ϕ in the homotopy class of the identity. Must ϕ be a homeomorphism?*

We conclude with the following conjecture related to the smooth classification of weakly hyperbolic higher rank lattice actions.

Conjecture V.9. *Let ρ be a C^∞ -smooth volume preserving, weakly hyperbolic action of a higher rank lattice Γ on a nilmanifold N/Λ . Then ρ is smoothly conjugate to an affine algebraic action ρ_0 on N/Λ .*

Perhaps a positive answer to Conjecture V.6 could be combined together with the normal form theory of Guysinsky, Katok, and Spatzier ([GK98], [KS97]) as an approach. An important first step would be to identify the representation coming

from Lemma IV.3 with the derivative representation coming from Zimmer's cocycle superrigidity theorem.

BIBLIOGRAPHY

BIBLIOGRAPHY

- [Ano67] D. V. Anosov, *Geodesic flows on closed riemannian manifolds of negative curvature.*, Proc. Steklov. Inst. Math. **90** (1967).
- [AS67] D.V. Anosov and Ya.G. Sinai, *Some smooth ergodic systems.*, Russian Math. Surveys **22:5** (1967), 103–168.
- [BG04] V. Bergelson and A. Gorodnik, *Modern dynamical systems and applications*, DMV Seminar, ch. Weakly mixing group actions: a brief survey and an example., pp. 81–97, Cambridge University Press, New York, 2004.
- [Bri95] M. Brin, *Lectures on spaces of nonpositive curvature*, DMV Seminar, ch. Appendix: Ergodicity of Geodesic Flows, pp. 81–97, Birkhauser Verlag, 1995.
- [Bri03] ———, *On dynamical coherence.*, Ergodic Theory Dynam. Systems (2003), 395–401.
- [Bro80] K. Brown, *Cohomology of groups.*, Springer-Verlag, 1980.
- [Del77] P. Delorme., *1-cohomologie des representations unitaires des groupes de lie semi-simples et resolubles. produits tensoriels continus representations.*, Bull. Soc. Math. France **105** (1977), 281–336.
- [DF04] K. Whyte. D. Fisher, D. Morris, *Non-ergodic actions, cocycles and superrigidity*, New York Journal Math. **10** (2004), 249–269.
- [dlH89] P. de la Harpe, *La propriete (t) de kazhdan pour les groupes localement compacts.*, Soc. Math. France, 1989.
- [dlL01] R. de la Llave., *Remarks on sobolev regularity in anosov systems.*, Ergodic Theory and Dynam. Systems **21** (2001), 1139–1180.
- [Fer80] R. Feres, *Dynamical systems and semisimple groups, an introduction*, Cambridge University Press, 1980.
- [Fis] D. Fisher, *Local rigidity of group actions: Past, present, future.*, preprint.
- [FM] D. Fisher and G. Margulis, *Local rigidity for affine actions of higher rank lie groups and their lattices.*, preprint.
- [Fra70] J. Franks, *Anosov diffeomorphisms*, Global Analysis, Proceedings of the Symposia in Pure Mathematics, vol. 14, American Mathematical Society, 1970, pp. 61–93.
- [GK98] M. Guysinsky and A. Katok, *Normal forms and invariant geometric structures on transverse contracting foliations.*, Math. Res. Lett. **5** (1998), 149–163.
- [Gui80] A. Guichardet, *Cohomologie des groupes topologiques et des algebres de lie.*, F. Nathan-Cedic (1980).
- [HK90] S. Hurder and A. Katok., *Differentiability, rigidity, and godbillon-vey classes for anosov flows.*, Publ. Math. Inst. Hautes Etudes Sci. **72** (1990), 5–61.

- [Hop39] E. Hopf, *Statistik der geodatischen linien in mannigfaltigkeiten negativer krummung*, Berichte Verh. Sachs Akad. Wiss Leipzig **91** (1939), 261–304.
- [Hop40] ———, *Statistik der losungen geodatischer probleme vom unstabilen typus. ii.*, Math. Ann. **117** (1940), 590–608.
- [Hur92] S. Hurder, *Rigidity for anosov actions of higher rank lattices.*, Ann. of Math. **135** (1992), 361–410.
- [Jou88] J.-L. Journé., *A regularity result lemma for functions of several variables.*, Revista Matematica Iberoamericana **4** (1988), 187–193.
- [KL91] A. Katok and J. Lewis., *Local rigidity for certain groups of toral automorphisms.*, Israel J. Math. **75** (1991), 203–241.
- [KP94] W. Kryszewski and S. Plaskacz., *Topological methods for the local controllability of non-linear systems.*, SIAM J. Control and Optimization **32** (1994), 213–223.
- [KS97] A. Katok and R. Spatzier, *Differential rigidity of anosov actions of higher rank abelian groups and algebraic lattice actions.*, Tr. Math. Inst. Steklova **216** (1997), 292–319.
- [Liv72] A. Livsic, *Cohomology of dynamical systems*, Nauk SSSR Ser. Mat. **36** (1972), 1296–1320.
- [Man74] A. Manning., *There are no new anosov diffeomorphisms on tori.*, Amer. J. Math. **96** (1974), 422–429.
- [Mar91] G. Margulis., *Discrete subgroups of semisimple lie groups.*, Springer, New York, 1991.
- [Mar99] ———, *Mathematics: Frontiers and perspectives*, ch. Problems and conjectures in rigidity theory., AMS, 1999.
- [MH77] M. Shub. M. Hirsch, C. Pugh, *Invariant manifolds*, Springer, 1977.
- [Mil97] J. Milnor, *Fubini foiled: Katok’s paradoxical example in measure theory.*, Math. Intelligencer **19** (1997), 30–32.
- [MQ01] G. Margulis and N. Qian., *Rigidity of weakly hyperbolic actions of higher real rank semisimple lie groups and their lattices.*, Ergod. Th. and Dynam. Sys. **21** (2001), 121–164.
- [NT98] V. Nitica and A. Torok., *Regularity of the transfer map for cohomologous cocycles.*, Ergod. Th. and Dynam. Sys. **18** (1998), 1187–1209.
- [OW98] D. Ornstein and B. Weiss., *On the bernoulli nature of systems with some hyperbolic structure.*, Ergod. Th. and Dynam. Sys. **18** (1998), 441–456.
- [Pes73] Ya. B. Pesin., *On the existence of invariant fibering for the diffeomorphisms of a smooth manifold.*, Math. USSR Sbornik **20** (1973), 213–222.
- [PS72] C. Pugh and M. Shub., *Ergodicity of anosov actions.*, Invent. Math. **15** (1972), 1–23.
- [PS00] ———, *Stable ergodicity and julienne quasiconformality.*, J. Eur. Math. Soc. **2** (2000), 1–52.
- [Rag66] M. Raghunathan, *Vanishing theorems for cohomology groups associated to discrete subgroups of semisimple lie groups.*, Osaka J. Math **3** (1966), 243–256.
- [Rag72] ———, *Discrete subgroups of lie groups.*, Springer-Verlag, 1972.
- [RdlL86] R. Moryion. R. de la Llave, J. M. Marco, *Canonical perturbation theory of anosov systems and regularity results for livsic cohomology equation.*, Ann. Math. **123** (1986), 537–612.

- [RT05] J. Rauch and M. Taylor, *Regularity of functions smooth along foliations, and elliptic regularity.*, Journal of Functional Analysis (2005), 74–93.
- [Sma67] Smale, *Differentiable dynamical systems.*, Bulletin of the AMS **73** (1967), 747–817.
- [Sta02] A.N. Starkov, *Vanishing of the first cohomologies for lattices in lie groups.*, Journal of Lie Theory **12** (2002), 449–460.
- [Tay96] M. Taylor., *Partial differential equations iii.*, Springer-Verlag, New York, 1996.
- [Wal75] P. Walters, *An introduction to ergodic theory*, Springer-Verlag, New York, 1975.
- [WM94a] D. Witte-Morris., *Arithmetic groups of higher \mathbb{Q} -rank cannot act on 1-manifolds.*, Proc. Amer. Math. Soc. **122** (1994), 330–340.
- [WM94b] ———, *Measurable quotients of unipotent translations on homogeneous spaces.*, Trans. Amer. Math. Soc. **354** (1994), 577–594.
- [Zim80] R.J. Zimmer., *Strong rigidity for ergodic actions of semisimple lie groups.*, Annals of Mathematics **112** (1980), 511–529.
- [Zim86] ———, *Actions of semisimple groups and discrete subgroups*, Proc. Internat. Cong. Math (1986), 1247–1258.
- [Zim90] ———, *Essential results of functional analysis*, Chicago Lectures in Mathematics, University of Chicago Press, 1990.

ABSTRACT

Weakly Hyperbolic Group Actions

by

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In this thesis, we investigate a class of dynamical systems, the weakly hyperbolic group actions, that arise naturally in the study of higher rank lattice actions. Motivated by classical results from the theory of hyperbolic dynamical systems, we provide results of two sorts. First, we establish the ergodicity of weakly hyperbolic group actions, generalizing Anosov's classical theorem for Anosov flows and diffeomorphisms. Secondly, for weakly hyperbolic group actions of lattice subgroups of higher rank noncompact simple Lie groups on tori, we show that weak hyperbolicity persists in the induced action on first homology. This result is a natural analogue to Manning's theorem concerning the classification of Anosov diffeomorphisms on tori.