AN *F*-SPACE WITH TRIVIAL DUAL AND NON-TRIVIAL COMPACT ENDOMORPHISMS

BY

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ABSTRACT

We give an example of an *F*-space which has non-trivial compact endomorphisms, but does not have any non-trivial continuous linear functionals.

1. Introduction

The object of this paper is to give an example of an *F*-space (complete, metrizable linear topological space) which has non-trivial compact endomorphisms but does not have non-trivial continuous linear functionals. An additional curious property of this space is that its algebra of continuous endomorphisms is not transitive; that is, there exist non-zero vectors f and g in the space such that no continuous endomorphism takes f to g.

In the opposite direction, D. Pallaschke [6] and P. Turpin [9] have recently shown that certain *F*-spaces of measurable functions already known to have trivial duals, in particular the spaces $L^p([0,1]$ for $0 < p < 1$, have no non-trivial compact endomorphisms.

Our example is constructed from the classical Hardy space H^p of analytic functions $(0 < p < 1)$, and relies heavily on the existence of certain rather explicitly determined proper, closed, weakly dense subspaces recently discovered by P. L. Duren, B. W. Romberg, and A. L. Shields [2]. The necessary background material is outlined in the next section, after which the example is constructed.

We wish to thank Professor A. Pełczyński for posing to us the question of whether such *F*-spaces could exist. In addition we are indebted to Professors David A. Gregory and Peter D. Taylor for a number of valuable conversations.

2. Preliminaries on H^p

A good reference for the material in this section is Duren's book [1], especially chapters 2 and 7. In what follows, Δ denotes the open unit disc in the complex plane. For $0 < p < \infty$ *the Hardy space H^p* is the collection of functions f analytic in Δ for which

$$\|f\|_p = \sup_{0 \leq r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right\}^{1/p} < \infty.$$

H^p is a linear space over the complex numbers, and if $1 \leq p < \infty$ then $\| \cdot \|_p$ is a norm which makes it into a Banach space. For $0 < p < 1$, the case of interest to us, the p-homogeneous functional $\| \cdot \|_p^p$ is subadditive and induces a translation-invariant metric

$$d(f, g) = \|f - g\|_p^p$$

on H^p which makes it into an F-space [1, p. 37, corollary 2].

When $0 < p < 1$ the functional $\| \cdot \|_p^p$ is not homogeneous, and the topology it induces on H^p is not locally convex [5]. We will nevertheless refer to $\| \cdot \|_p^p$ as the *norm* on H^p, and call the corresponding topology the *norm topology*. Note that the positive multiples of the unit ball

$$\{ f \in H^p : \quad \|f\|_p^p \leq 1 \}$$

form a local base for the norm topology; and therefore a subset B of H^p is (topologically) bounded if and only if it is *norm bounded*:

$$\sup\{\|f\|_p^p : \quad f \in B\} < \infty.$$

PROPOSITION 2.1. *Every bounded subset of H^p is a normal family $(0 < p < \infty)$.*

PROOF. We have the estimate

$$|f(z)| \leq 2^{1/p} \|f\|_p (1 - |z|)^{-1/p} \qquad (z \in \Delta)$$

for $f \in H^p$ [1, p. 36], so if B is a bounded subset of H^p, then the members of B are bounded uniformly on compact subsets of Δ. It follows that B is a normal family [7, theorem 14.6, p. 272].

Let κ denote the restriction to H^p of the topology of uniform convergence on compact subsets of Δ. It is well known that κ is locally convex and metrizable.

PROPOSITION 2.2.   *The closed unit ball of H^p is κ-compact $(0 < p < \infty)$.*

PROOF.   Since κ is metrizable it is enough to show that each sequence in the closed unit ball U of H^p has a subsequence κ-convergent to an element of U. Suppose (f_n) is a sequence in U. Since U is a normal family (Proposition 2.1) there is subsequence (f_{n_j}) and an analytic function f on Δ such that $f = \kappa - \lim f_{n_j}$. For $0 \le r < 1$ the sequence (f_{n_j}) converges to f uniformly on the circle $|z| = r$, hence

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \ = \lim_{j \to \infty} \frac{1}{2\pi} \int_0^{2\pi} |f_{n_j}(re^{it})|^p \, dt \ \le 1,$$

from which it follows that $\|f\|_p \le 1$. Thus $f \in U$ and the proof is complete.

One of the most important facts about H^p spaces is that for each f in H^p the *radial limit*

$$f^*(e^{it}) = \lim_{r \to 1-} f(re^{it})$$

exists for almost every real t [1, theorem 2.2, p. 17], and moreover

(2.1)
$$\|f\|_p^p = \frac{1}{2\pi} \int_0^{2\pi} |f^*(e^{it})|^p \, dt$$

[1, theorem 2.6, p. 21]. A function q analytic in Δ is called an *inner function* if $|q| \le 1$ on Δ, and $q^*(e^{it}) = 1$ a.e. It follows from Eq. (2.1) that for q inner the multiplication map $f \to qf$ is an isometry on H^p, so its range qH^p is a closed subspace. If, moreover, q is *non-trivial* (i.e. $q \ne 1$), then the Canonical Factorization Theorem [1, theorem 2.8, p. 24] shows that qH^p is a *proper* subspace of H^p. We can now state a result of Duren, Romberg, and Shields which provides the key to our example.

THEOREM 2.3.   [2, theorem 13, p. 53]. *There exists a non-trivial inner function q such that qH^p is dense in weak topology of H^p for $0 < p < 1$.*

We remark that every non-locally convex F-space has a closed subspace that is not weakly closed. By [2, theorems 16 and 17, p. 59] an equivalent assertion is that the extension form of the Hahn-Banach theorem fails in every non-locally convex F-space; and Kalton [3, corollary 5.3] has recently proved this latter assertion. We do not know if every non-locally convex F-space must have a proper, closed, weakly dense subspace. According to Theorem 2.3, such subspaces exist in H^p $(0 < p < 1)$. They have also been found in certain other F-spaces of analytic functions, as well as in l^p $(0 < p < 1)$ [8].

3.   The example

Suppose E and F are linear topological spaces. A linear transformation $T : E \rightarrow F$ is said to be *compact* (or *completely continuous*) if there is a neighborhood of 0 in E whose image under T is compact in F. It is easy to see that every compact linear transformation is continuous, and that the composition of a compact and a continuous linear transformation (in either order) is again compact.

The collection of continuous linear functionals on E is called the *dual* of E, denoted by E'. If $E' = \{0\}$ we say E has *trivial dual*. An *endomorphism* of E is a linear transformation of E into itself. We now state our main result.

THEOREM 3.1.   *There is an F-space with trivial dual which has non-trivial compact endomorphisms.*

The proof will require some preliminaries. In particular we need a certain topology on H^p intermediate between κ (the topology of uniform convergence on compact subsets of Δ) and the norm topology. This topology, denoted by β, is *the strongest topology on H^p that agrees with κ on every norm bounded subset*. It is easy to see that such a topology exists: we simply declare a set to be β-open (respectively β-closed) if its intersection with every bounded set B is relatively κ-open (respectively κ-closed) in B. It is easy to see that these "β-open" sets actually satisfy the axioms for a topology. Since a subset of H^p is bounded if and only if it is norm bounded, if we wish to decide whether a set is β-open or β-closed, then we need only consider its intersection with every positive (or even positive integer) multiple of the closed unit ball.

Since β is stronger than the Hausdorff topology κ, it is itself Hausdorff. Since the closed unit ball of H^p is κ-compact (Proposition 2.1), it is also β-compact. Finally, β is weaker than the-norm topology of H^p, since it is weaker on bounded sets. To summarize:

PROPOSITION 3.2. β is a Hausdorff topology on H^p intermediate between κ and the norm topology. The closed unit ball of H^p is β-compact $(0 < p < \infty)$.

It will be important for us to known that β is actually a vector topology — a fact we have not assumed in advance. We give a proof modelled after that of the Banach-Dieudonné theorem [4, sec. 2.2, pp. 211–212].

PROPOSITION 3.3. β is a vector topology.

PROOF. For brevity we will refer to a κ-closed κ-neighborhood of zero as an *admissible* neighborhood. We denote the closed unit ball of H^p by U. It is easy to check that the collection of all sets of the form

$$(3.1) \qquad \bigcap_{n=1}^{\infty} (p_n U + V_n)$$

where (p_n) is a real sequence with $0 \le p_n \to \infty$, and (V_n) is a sequence of admissible neighborhoods, is a local base for a vector topology β' on H^p (in fact, it follows from the work of Wiweger [10, sec. 2.3, p. 52], that β' is the strongest vector topology on H^p agreeing with κ on bounded sets). We are going to show that $\beta' = \beta$.

To see that $\beta' \le \beta$, suppose B is a bounded subset of H^p, so $B \subseteq kU$ for some $k > 0$. Suppose N is a β'-neighborhood of zero of the form (3.1). Then whenever $p_n \ge k$ we have

$$p_n U + V_n \supseteq kU \supseteq B,$$

so

$$B \cap N = B \cap \bigcap_{p_n \le k} (p_n U + V_n) = B \cap W$$

where

$$W = \bigcap_{p_n \le k} (p_n U + V_n)$$

is a κ-neighborhood of zero. Thus β' agrees with κ on bounded sets, so $\beta' \le \beta$.

In the other direction, suppose A is a β-open set containing the origin. We claim that there exists a sequence (V_k) of admisible neighborhoods such that for each integer $n \ge 1$:

$$(3.2) \qquad nU \cap \bigcap_{k=1}^{n} [(k-1)U + V_k] \subseteq A.$$

Then the β'-neighborhood of zero

$$\bigcap_{k=1}^{\infty} [(k-1)U + V_k]$$

will be contained in A, completing the proof.

We obtain the sequence (V_k) by induction. Since $\beta = \kappa$ on U we know there is an admissible neighborhood V_1 such that

$$U \cap V_1 \subsetneq U \cap A \subseteq A,$$

and this is just inequality (3.2) for $n = 1$. So suppose $V_1, \cdots, V_n$ are admissible neighborhoods satisfying (3.2). We want to find V_{n+1} so that $V_1, \cdots, V_{n+1}$ also satisfy (3.2). Suppose we cannot; that is, suppose

$$(3.3) \qquad (n+1)U \cap \bigcap_{k=1}^{n} [(k-1)U + V_k] \cap (nU + V) \cap A^c \neq \phi$$

for each admissible neighborhood V (here A^c denotes the complement of A in H^p). Since A is β-open and $(n+1)U$ is β-compact, the set $(n+1)U \cap A^c$ is β-compact, hence κ-compact. It follows from the κ-compactness of U that $\alpha U + V$ is κ-closed for every admissible neighborhood V. Thus the left side of (3.3) is, for each admissible V, a non-void κ-closed subset of the κ-compact space $(n+1)U$. It follows easily from (3.3) that the family of all these left sides has the finite intersection property, and hence a common point: that is,

$$(3.4) \qquad (n+1)U \cap \bigcap_{k=1}^{n} [(k-1)U + V_n] \cap \bigcap_{V} (nU + V) \cap A^c \neq \phi,$$

where V ranges over all admissible neighborhoods. Now the κ-compactness of nU guarantees that $\bigcap_V (nU + V) = nU$ [4; theorem 5.2 (v), p. 35], so (3.4) reduces to

$$nU \cap \bigcap_{k=1}^{n} [(k-1)U + V_k] \cap A^c \neq \phi,$$

which contradicts (3.2). This completes the proof.

We note in passing that this proof uses only the fact that H^p is a Hausdorff

locally bounded linear topological space, and κ is a Hausdorff vector topology for which each norm bounded set is relatively compact. That is, we have really proved:

PROPOSITION 3.3'.   *Suppose E is a Hausdorff locally bounded linear topological space and κ is a Hausdorff vector topology on E for which each norm bounded subset of E is relatively compact. Then the strongest topology that agrees with κ on each norm bounded set is actually a vector topology.*

Finally we need to know that certain norm-closed subspaces of H^p are also β-closed.

PROPOSITION 3.4.   *For every inner function q the subspace qH^p is β-closed in* H^p.

PROOF.   Let U denote the closed unit ball of H^p. It is enough to show that $U \cap qH^p$ is κ-closed in H^p. So suppose (f_n) is a sequence in $U \cap qH^p$, $f \in H^p$, and $f = \kappa - \lim f_n$. Now there exists a sequence (h_n) in H^p such that $f_n = qh_n$ for each n; and since

$$\| h_n \|_p = \| f_n \|_p \leq 1$$

we have $(h_n) \subseteq U$. Since U is κ-compact (Proposition 2.1) there is a subsequence (h_{n_j}) which is κ-convergent to some h in U. Consequently

$$qh = \kappa - \lim qh_{n_j} = \kappa - \lim f_{n_j} = f,$$

so $f \in U \cap qH^p$, and the proof is complete.

PROOF OF THEOREM 3.1.   Fix $0 < p < 1$. By Theorem 2.3 we can choose a non-trivial inner function q such that the (proper) closed subspace qH^p is weakly dense. Let E_N denote the quotient linear topological space H^p/qH^p, where H^p has its norm topology. Since qH^p is norm closed in H^p, E_N is Hausdorff: in fact it is complete and metrizable. Since qH^p is weakly dense in H^p, the quotient space E_N has trivial dual [2, corollary 1, p. 53].

Let E_β denote the quotient linear topological space H^p/qH^p, *where H^p has the β-topology. E_β* is Hausdorff since qH^p is β-closed (Proposition 3.4). Now E_N and E_β are the same linear space H^p/qH^p, but with different topologies.   It is

not difficult to see that the topology of E_β is weaker than that of E_N, since the β-topology is weaker on H^p than the norm topology. In particular the identity map $j_{N,\beta}: E_N \to E_\beta$ is continuous, and E_β also has trivial dual.

Let U denote the closed unit ball of H^p and let π denote the quotient map taking H^p onto H^p/qH^p. Then π is continuous when viewed as a map of H^p in its norm topology onto E_N, and also when viewed as a map of H^p in the β-topology onto E_β. In addition, $V = \pi(U)$ is the closed unit ball of E_N, and it is a compact subset of E_β, since U is β-compact in H^p. Thus the identity map $j_{N,\beta}: E_N \to E_\beta$ takes the closed unit ball of E_N onto a compact subset of E_β, and is therefore a compact linear transformation.

Now any vector topology can be represented as the least upper bound of a family of pseudo-metric topologies [4, section 6, problem C, p. 51–52]. In particular there is a pseudo-metric d on E_β that induces a (not necessarily Hausdorff) vector topology weaker than β, yet different from the indiscrete topology. Let E_d denote E_β equipped with this new topology: then the identity map $j_{\beta,d}: E_\beta \to E_d$ is continuous. Let F denote the closure of $\{0\}$ in E_d. Then F is a proper, closed subspace of E_d: and it is not difficult to see that the quotient space E_d/F is a non-trivial, *metrizable* linear topological space.

Since the quotient map $\rho: E_d \to E_d/F$ is continuous, so is its composition with $j_{\beta,d}$. Recall that the identity map $j_{N,\beta}: E_N \to E_\beta$ is compact; thus the composition $S = \rho \circ j_{\beta,d} \circ j_{N,\beta}$ is a non-trivial compact linear map taking E_N onto the (necessarily incomplete) linear metric space E_d/F. Let E_M be the completion of E_d/F. Then E_M is an F-space, and S can be regarded as a non-trivial compact linear transformation from E_N into E_M. Let $E = E_N \oplus E_M$ and define $T: E \to E$ by

$$T(x, y) = (0, Sx) \qquad (x \in E_N, \ y \in E_M).$$

Then E is an F-space since E_N and E_M are, and T is a non-trivial compact endomorphism of E.

It remains to show that E has non-trivial dual. We have already observed that E_N has trivial dual, as does E_β. Since E_d is just E_β in a weaker topology, it too has trivial dual, as does its quotient E_d/F. Thus E_M, which is the completion of E_d/F also has trivial dual, hence so does $E = E_N \oplus E_M$. This completes the proof.

The space E that we have constructed has a further curious property. A linear topological space is said to be *transitive* if for each pair f, g of non-zero vectors there is a continuous endomorphism of the space which takes f to g. For

example, it is easy to see that any linear topological space whose dual separates points is transitive; while the direct sum of a space with trivial dual and its scalar field is not transitive. Pełczyński has observed that a transitive linear topological space with non-trivial compact endomorphisms must also have non-trivial dual. This result is stated and proved in [6, theorem 1.2, p. 125] for real scalars. The essential feature of the proof is an application of the Riesz theory of compact operators, which holds as well for complex scalars [4, chapter 5, problems A and B, pp. 206–207). So Pełczyński's result and its proof as given in [6] hold in the complex case, and we obtain from it and Theorem 3.1 the following:

COROLLARY. *There exists a non-transitive F-space with trivial dual.*

We note in closing that the space E constructed in Theorem 3.1 can be chosen to be locally bounded. To see this, let us fix $0 < p < 1$ and revert to the notation used in the proof of Theorem 3.1. Since the topology β on H^p has a local base of absolutely p-convex sets, so does the quotient space E_β (a subset S of a real or complex linear space is *absolutely p-convex* if $ax + by \in S$ whenever $x, y \in S$ and $|a|^p + |b| \le 1$). The "Minkowski functional" (cf. [4, p. 15]) of such a neighborhood is subadditive and absolutely p-homogeneous, and one of these functionals can be used to induce the pseudo-metric d. It follows quickly that the metric on E_M is also induced by a subadditive, p-homogeneous functional, hence E_M is locally bounded. Now E_N is locally bounded by the definition of the quotient metric, hence so is the direct sum $E = E_M \oplus E_N$, and our assertion is proved.

Added December 2, 1974. After this paper was submitted P. Turpin pointed out to us that Proposition 3.3′ is a special case of a result of L. Waelbroeck (*Topological Vector Spaces and Algebras*, Springer Lecture Notes in Mathematics, No 230, Proposition 6.2, p. 48).

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