The Cyclic Behavior of Translation Operators on Hilbert Spaces of Entire Functions

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Dedicated to the memory of Domingo Herrero

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Abstract. We show that the translation operator $T : f(z) \to f(z+1)$, acting on certain Hilbert spaces consisting of entire functions of slow growth, is *hypercyclic* in the sense that for some function f in the space, the orbit $\{T^n f\}_0^\infty$ is dense. We further show that the operator $T - I$ can be made compact, with approximation numbers decreasing as quickly as desired, simply by choosing the underlying Hilbert space to be sufficiently small. This shows that hypercyclic operators can arise as perturbations of the identity by "arbitrarily compact" operators. Our work extends that of G.D. Birkhoff (1929), who showed that T is hypercyclic on the Fréchet space of *all* entire functions, and it complements recent work of Herrero and Wang, who were the first to discover that perturbations of the identity by compacts could be hypercyclic.

Introduction

This paper originates from two sources, one old and one new. The old one is G.D. Birkhoff's intriguing observation about the orbits of translation operators acting on the space of entire functions [**Bir**, 1929]. To state this result precisely, we introduce the space $\mathcal{E}$ of entire functions of one complex variable, endowed with the topology of uniform convergence on compact subsets of the plane, and the operator $T_a : \mathcal{E} \to \mathcal{E}$ of "translation by the complex number a," defined by:

$$T_a f(z) = f(z+a) \qquad (f \in \mathcal{E}, \quad z \in \mathbb{C}).$$

Birkhoff's theorem asserts that *if $a \neq 0$, then there is a function $f \in \mathcal{E}$ whose orbit $\{T_a^n f\}_0^\infty$ is dense in $\mathcal{E}$.*

This result of Birkhoff can be viewed in several ways. Taken at face value it provides a "universal" entire function which, over any compact set, has translates that approximate any entire function as accurately as desired. In the language of

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dynamics, Birkhoff's theorem states that each translation operator is *topologically transitive* on $\mathcal{E}$, and this is the major step in proving that these operators are actually *chaotic* in one of the commonly accepted senses (see [**Dev**, page 50] and [**GoS**, Section 5]).

In this paper, however, we are going to concentrate on the connection that Birkhoff's theorem establishes between operator theory and complex analysis, with particular emphasis on the notion of *cyclicity.* Recall that an operator A on a Fréchet space is *cyclic* if there is a vector f in the space (the *cyclic vector* for A) whose orbit $\{A^n f\}_0^\infty$ has *dense linear span.* If the orbit itself is dense without any additional help from the linear span, we call the operator *hypercyclic*, and refer to f as a *hypercyclic vector* for A. Thus hypercyclicity is the strongest possible form of cyclicity, and Birkhoff's theorem states that *every translation operator* T_a $(a \neq 0)$ *is hypercyclic on* $\mathcal{E}$.

At first glance it might appear that the hypercyclic vectors promised by this result must grow rapidly in order to have translates that approximate every entire function, but this is not the case. Recently S.M. Duyos-Ruiz proved that given $a \neq 0$ and any preassigned transcendental growth rate, there is a hypercyclic cyclic vector for T_a with slower growth [**DuR**]. This result, in turn, suggests the possibility that translation operators might be hypercyclic on Fréchet spaces of entire functions defined by such slow-growth conditions (this is not implied by Duyos-Ruiz' theorem).

QUESTION 1. *Can Birkhoff's theorem be extended to Hilbert spaces of entire functions having "arbitrarily slow growth?"*

Our preference for Hilbert space arises from a desire to emphasize properties of operators over those of the underlying spaces. As the reader will see, our work could as well have been set in many different kinds of Fréchet spaces of entire functions.

We will also see that "larger" spaces of entire functions tend to pose fewer obstacles to hypercyclicity than "smaller" ones. There is, however, a limit that must be placed on our interpretation of the phrase "arbitrarily slow growth." *No finite dimensional Hilbert space supports a hypercyclic operator* [**Kit**]. In other words, polynomial growth conditions must be excluded. Thus Question 1 really asks if any *transcendental* growth condition must be similarly excluded.

The second, and more modern, source of our investigation is recent work of Herrero and Wang [**HerW**] implying that on Hilbert space the identity operator (that most non-cyclic of all operators) can be perturbed by a compact operator in such a way that the result is hypercyclic. This result is surprising because *no perturbation of the identity by a finite dimensional operator can be hypercyclic* (see section 4). These considerations suggest:

QUESTION 2. *On Hilbert space, can "arbitrarily compact" infinite rank operators perturb the identity to a hypercyclic operator?*

The result of Herrero and Wang follows from their characterization of the operator norm closure of the hypercyclics, but their proof seems too complicated to give any information about Question 2. In fact additional motivation for Question 2 comes from our desire to find naturally occurring examples of hypercyclic perturbations of the identity by compacts.

In this paper we initiate a program that answers both questions affirmatively (subject, of course, to proper interpretation of phrases like "arbitrarily compact"), and raises interesting problems for future consideration. First, we extend Birkhoff's theorem to a natural collection of Hilbert spaces of entire functions which can be "arbitrarily small" in the sense that the intersection of the whole collection is just the space of polynomials. Then we show that the operator $T_a - I$ is compact on each of these spaces, and that its compactness intensifies as the underlying space decreases in size. In fact, we show that the approximation numbers of $T_a - I$ can be made to tend to zero as quickly as desired, simply by requiring the underlying Hilbert space to consist of entire functions with suitably slow growth.

It is not difficult to see that we can also make the norm of $T_a - I$ as small as desired simply by choosing the translation vector a close enough to zero. Thus we answer the second question as follows: *The identity can be perturbed to be hypercyclic by operators which, subject only to the restriction of infinite rank, have arbitrarily small norm, and arbitrarily high degree of compactness.* For example, given $\epsilon > 0$, we can arrange for the perturbing operator $T_a - I$ to have norm $< \epsilon$, and belong to every Schatten p - class.

Organization of the paper. We wish to make our work readily accessible to specialists in both operator theory and complex analysis, so experts will occasionally find well known results in their field belabored here for the sake of expository completeness.

In section 1 we define the Hilbert spaces of entire functions in which our results are set, and develop some of their elementary properties. Section 2 establishes the hypercyclicity of translation. The main result (Theorem 2.1) can be viewed as a considerable strengthening of Duyos-Ruiz's theorem, with a proof that is made more transparent thanks to the organizational powers of functional analysis. In the third section we study the compactness of $T_a - I$. Here everything is related to the compactness of the differentiation operator, and in this regard our work is anticipated by that of V.A. Bogachev [**Bog**], who discussed the compactness of differentiation on suitably small Banach spaces of entire function defined by sup-norm growth conditions. The paper closes with a brief section devoted to related issues, references, and open problems.

Additional historical background. The modern study of hypercyclicity is a relatively new pursuit, but as we pointed out earlier, its roots extend back at least six decades. Thus it seems appropriate to take these last few introductory paragraphs to fill in some more of the historical context of our work.

Thirteen years after the publication of Birkhoff's theorem, Seidel and Walsh [**SW**] imported the result to the unit disc, replacing translation by certain conformal disc automorphisms. This idea has recently been developed in the Hilbert space context by Bourdon and Shapiro [**BoS1**] as part of an extensive program to classify the cyclic and hypercyclic behavior of composition operators on the Hardy space H^2. These authors have obtained complete results for linear fractional maps of the unit disc into itself, and using the linear fractional maps as "models" for more complicated ones, have extended this classification theorem to a very general class of composition operators [**BoS2**].

As for other operators, G.R. MacLane showed in 1952 that differentiation is also hypercyclic on the space of entire functions [**MacL**], and recently Godefroy and Shapiro filled in everything between the theorems of Birkhoff and MacLane by showing that: *Every continuous linear operator on $\mathcal{E}$ that commutes with differentiation, and is not a scalar multiple of the identity, is hypercyclic* [**GoS**, Theorem 5.1].

The study of hypercyclicity for Hilbert space operators originated with S. Rolewicz [**Rol**], who showed in 1969 that if B is the backward shift on ℓ^2, defined by

$$B(\{a_0, a_1, a_2, \dots\}) = \{a_1, a_2, a_3 \dots\},$$

then λB is hypercyclic for any complex number λ of modulus > 1 (note that B itself is a contraction on ℓ^2, so it cannot be hypercyclic without some additional help). In the work cited above, Godefroy and Shapiro placed Rolewicz's result in the context of adjoint multipliers on the Hardy space H^2 with this result: *if ϕ is a bounded holomorphic function on the open unit disc U, and M_ϕ denotes the associated operator of pointwise multiplication on H^2, then* the adjoint multiplier *$(M_\phi)^*$ is hypercyclic on H^2 if and only if $\phi(U)$ intersects the unit circle.* [**GoS**, section 4]. Rolewicz's operator λB corresponds to the adjoint multiplier induced by $\phi(z) = \lambda z$.

This result of Godefroy and Shapiro also shows that the adjoint multiplier induced on H^2 by the function $\phi(z) = 1 + \epsilon z$ is hypercyclic for each complex number ϵ. As $\epsilon \to 0$, these operators converge in norm to the identity, and therefore provide another concrete example of norm-approximation of the identity by hypercyclic operators. However, unlike the translation operators to be studied here, none of these adjoint multipliers differs from the identity by a compact.

All the above-mentioned results of Bourdon, Godefroy, and Shapiro depend on a sufficient condition for hypercyclicity that was proved originally by Carol Kitai in her Toronto dissertation [**Kit**, 1982], and rediscovered a few years later by Gethner and Shapiro [**GeS**]. A version of this sufficient condition (Proposition 2.2) plays a major role in this paper as well. Unfortunately, most of Kitai's thesis, including the sufficient condition for hypercyclicity, was never published. Her work contains many interesting results about general properties of hypercyclic operators, and we will discuss some of these in section 4.

1. Hilbert spaces of entire functions

In this section we describe the Hilbert spaces of entire functions in which the rest of our work is set, and record their most basic properties. Following Duyos-Ruiz [**DuR**], let us call an entire function $\gamma(z) = \sum \gamma_n z^n$ a *comparison function* if $\gamma_n > 0$ for each n, and the sequence of ratios γ_{n+1}/γ_n decreases to zero as n increases to ∞. For each comparison function γ we define $E^2(\gamma)$ to be the Hilbert space of power series

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

for which

$$\|f\|_{2,\gamma}^2 \equiv \sum_{n=0}^{\infty} \gamma_n^{-2} |\hat{f}(n)|^2 < \infty.$$

It is easy to check that each element of $E^2(\gamma)$ is an entire function, and that every sequence convergent in the norm of the space is uniformly convergent on compact subsets of the plane (see Proposition 1.4 below). We will be interested in the operators of translation and differentiation on these spaces. In what follows, T_a denotes the operator of "translation by a," as defined in the Introduction, and D denotes differentiation.

1.1 Proposition. *The operator D is bounded on $E^2(\gamma)$ if and only if the sequence $\{n\gamma_n/\gamma_{n-1}\}_1^\infty$ is bounded. D is compact on $E^2(\gamma)$ if and only if this sequence converges to zero.*

Proof: The functions

$$e_n(z) = \gamma_n z^n \qquad (n = 0, 1, 2, \dots)$$

form an orthonormal basis for $E^2(\gamma)$, relative to which the operator D is a *weighted backward shift:*

$$De_k = \begin{cases} w_k e_{k-1}, & \text{for } k > 0 \\ 0, & \text{for } k = 0, \end{cases}$$

with

$$w_k = \frac{k\gamma_k}{\gamma_{k-1}}. \tag{1}$$

With this observation, the proof becomes routine. Let $R = \sup\{w_k : k \geq 1\}$. If D is bounded, then for each positive integer k,

$$w_k = \|De_k\| \leq \|D\| \, \|e_k\| = \|D\|,$$

so $R < \infty$. Conversely, if $R < \infty$ then for every holomorphic polynomial f we have

$$Df = \sum w_{k+1} < f, e_{k+1} > e_k$$

where the sum has only finitely many terms, and "$<,>$" denotes the inner product of $E^2(\gamma)$. Thus

$$\|Df\|^2 = \sum w_{k+1}^2 | < f, e_{k+1} > |^2 \leq R^2 \|f\|^2.$$

Since the polynomials are dense in $E^2(\gamma)$, it follows immediately that D is bounded on $E^2(\gamma)$, with norm $\leq R$. (Note that the two halves of this argument prove that $\|D\| = R$.)

As for compactness, suppose first that $w_k \to 0$. Fix a positive integer n, and let D_n be the operator of rank n defined by:

$$D_n f = \sum_{k=0}^{n-1} w_{k+1} < f, e_{k+1} > e_k \qquad (f \in E^2(\gamma)).$$

Then for each $f \in E^2(\gamma)$,

$$\|(D - D_n)f\| = \left\| \sum_{k=n}^{\infty} w_{k+1} < f, e_{k+1} > e_k \right\| \leq \sup_{k \geq n} w_{k+1} \|f\|,$$

hence

$$\|D - D_n\| \leq \sup_{k \geq n} w_{k+1} \to 0.$$

Thus D is the limit in norm of a sequence of finite rank operators, so it is compact.

Conversely, if D is compact on $E^2(\gamma)$ then, since $e_n \to 0$ weakly, we have $w_n = \|De_n\| \to 0$. ∎

Our interest in the differentiation operator stems from the intimate connection with translation that is exhibited by the following corollary.

1.2 COROLLARY. *Suppose the sequence $\{n\gamma_n/\gamma_{n-1}\}$ is bounded. Then each translation operator T_a is bounded on $E^2(\gamma)$, and*

$$T_a = \sum_{n=0}^{\infty} \frac{a^n}{n!} D^n, \tag{2}$$

where the series on the right converges in the operator norm.

PROOF: It is well known, and not difficult to show, that (2) holds for the full space $\mathcal{E}$ of entire functions, in the sense that when each term of the series on the right is applied to a function $f \in \mathcal{E}$, the result converges uniformly on compact subsets of the plane to the function $f(z+a)$. One way to see this is to note that the Cauchy inequalities yield estimates proving that the series on the right converges pointwise on $\mathcal{E}$ to a continuous linear operator, which one easily checks agrees with T_a on the collection of exponentials $e^{\lambda z}$ (cf. [**GoS**, Proposition 5.2]). That the two operators coincide on all of $\mathcal{E}$ follows from the fact that these exponentials span a dense linear subspace of $\mathcal{E}$.

Once we know this, it only remains to note that since D is bounded (Proposition 1.1), the series on the right side of (2) converges in operator norm to a bounded operator on $E^2(\gamma)$, and since convergence in the space $E^2(\gamma)$ implies convergence in $\mathcal{E}$ (see Proposition 1.4), this bounded operator must be T_a. ∎

Remark: We might paraphrase this last result as follows: *$T_a = e^{aD}$ whenever D is bounded on $E^2(\gamma)$.* The proof yields more; it shows that if $\Phi(z) = \sum a_n z^n$ is a function holomorphic in a neighborhood of the closed disc $\{|z| \leq \|D\|\}$, then the series $\sum a_n D^n$ converges in the operator norm of $E^2(\gamma)$ to a bounded linear operator on $E^2(\gamma)$, which we might reasonably denote by $\Phi(D)$. The real story goes deeper, and to explore it efficiently we need to focus some more attention on the properties of those comparison functions that render the differentiation operator bounded.

Definition. *If the sequence $w_n = n\gamma_n/\gamma_{n-1}$ is monotonically decreasing, we call γ an* admissible *comparison function.*

Equivalently: *γ is an admissible comparison function precisely when the sequence $\{\log(n!\gamma_n)\}$ is concave.* This contrasts with the definition of comparison function, which merely requires that $\{\log \gamma_n\}$ be concave.

Since we are going to focus on small spaces, the following elementary facts, whose proofs we leave to the reader, assert that nothing will be lost by considering only those spaces $E^2(\gamma)$ where γ is an admissible comparison function. In particular, the last statement asserts that in some sense our spaces can be "arbitrarily small."

(1) *If D is bounded on $E^2(\gamma)$, then there is an admissible comparison function $\tilde{\gamma}$ such that $\tilde{\gamma}_n \leq \gamma_n$ for each n.*

(2) *The intersection of all the spaces $E^2(\gamma)$, as γ runs through all admissible comparison functions, is precisely the collection of holomorphic polynomials.*

Although its full strength is not required for the sequel, the next result provides some useful perspective on the spaces $E^2(\gamma)$.

1.3 PROPOSITION. *Suppose γ is an admissible comparison function. Let $\tau = \lim n\gamma_n/\gamma_{n-1}$. Then:*

(a) The entire function γ is of order 1 and type τ.

(b) The spectrum of the operator D is the closed disc $\tau\bar{U} = \{|z| \leq \tau\}$.

(c) If $\Phi(z) = \sum a_n z^n$ is a function holomorphic in a neighborhood of $\tau\bar{U}$, then the series $\sum a_n D^n$ converges in the operator norm of $E^2(\gamma)$ to a bounded linear operator (which we henceforth refer to as $\Phi(D)$) on $E^2(\gamma)$.

PROOF: (a) Fix $\sigma > \tau$. We are assuming that $n\gamma_n/\gamma_{n-1} \searrow \tau$, so for some $N > 0$ we have

$$\gamma_n < \frac{\sigma}{n}\gamma_{n-1} \qquad (n \geq N).$$

Thus for a constant C that depends only on σ we have

$$\gamma_n < C\frac{\sigma^n}{n!} \qquad (n = 0, 1, \dots),$$

from which it follows that

$$|\gamma(z)| \leq \sum \gamma_n |z|^n \leq C \sum \frac{(\sigma|z|)^n}{n!} = Ce^{\sigma|z|}$$

for each $z \in \mathrm{C}$. Thus γ is of order 1 and type $\leq \tau$.

In the other direction, we have $n\gamma_n/\gamma_{n-1} \geq \tau$ for each n, hence

$$(*) \qquad \gamma_n \geq \tau^n \gamma_0/n! \qquad (n = 0, 1, 2, \cdots).$$

Thus for $r \geq 0$,

$$\gamma(r) \geq \gamma_0 e^{\tau r},$$

so γ is of type exactly τ.

(b) A computation of the sort used to prove Proposition 1.1 shows that for each non-negative integer k,

$$\|D^k\| = \sup_{n \geq 1} w_n w_{n+1} \cdots w_{n+k-1} = w_1 w_2 \cdots \omega_k,$$

where $w_n = n\gamma_n/\gamma_{n-1}$, and the last equality is due to the fact that $\{w_n\}$ is assumed to be a decreasing sequence. Thus

$$\|D^k\|^{1/k} = (w_1 w_2 \cdots \omega_k)^{1/k} = \left(\frac{k!\gamma_k}{\gamma_0}\right)^{1/k},$$

so the spectral radius of D is

$$\rho(D) = \lim \|D^k\|^{1/k} = \lim(k!\gamma_k)^{1/k}.$$

This limit of successive roots of the sequence $\{k!\gamma_k\}$ coincides with the corresponding limit of successive ratios (see, e.g., [**Rud1**, Theorem 3.37]), which by definition is just τ.

So $\rho(D) = \tau$, hence $\sigma(D) \subset \tau\bar{U}$. To obtain equality we need only observe that, by (*) above, whenever $|\lambda| < \tau$, the exponential function $e^{\lambda z}$ belongs to $E^2(\gamma)$. Since this function is an eigenvector of D with corresponding eigenvalue λ, we see that $\tau U \subset \sigma(D)$. This proves that $\sigma(D) = \tau\bar{U}$.

(c) This follows immediately from the fact, just derived in the course of proving part (b) above, that $\|D^k\|^{1/k} \to \tau$. ∎

REMARK: In the sections to follow we will focus on spaces $E^2(\gamma)$ for which γ is admissible, and on which D is *compact*. In view of Propositions 1.1 and 1.3 this means we will be interested primarily in admissible comparison functions of *exponential type zero*. On the corresponding spaces $E^2(\gamma)$, Proposition 1.3 asserts that D will be *quasinilpotent* (its spectrum is the singleton $\{0\}$), and that $\Phi(D)$ can be defined for every function Φ that is holomorphic in a neighborhood of the origin. In such a case, the spectrum of $\Phi(D)$ is the singleton $\{\Phi(0)\}$. This just restates the fact that $\Phi(D)$ is invertible if and only if $\Phi(0) \neq 0$, and this in turn reflects the corresponding invertibility property of functions holomorphic in a neighborhood of zero.

It will be important for us to know how restrictions on the growth of comparison functions are reflected in the behavior of the functions in the corresponding Hilbert spaces. This is most naturally done by comparing the norm of $E^2(\gamma)$ with a Banach space norm that is more directly related to functional values. For each comparison function γ, and each entire function f, we define the possibly infinite "norm"

$$\|f\|_{\infty,\gamma} = \sup\{|f(z)|\gamma(|z|)^{-1} : z \in \mathrm{C}\},$$

and set

$$E^\infty(\gamma) = \{f \in \mathcal{E} : \|f\|_{\infty,\gamma} < \infty\}.$$

Thus $E^\infty(\gamma)$ is the space of all entire functions dominated by constant multiples of γ. When endowed with the norm $\| \ \|_{\infty,\gamma}$ it is a Banach space for which one could also study the properties of differentiation and translation. As we mentioned earlier, Bogachev [Bog] has studied the problem of compactness of D on such spaces.

The next result says that the spaces $E^\infty(\gamma)$ and $E^2(\gamma)$ are "almost the same," and that their norms are "almost equivalent."

1.4 PROPOSITION. *(a) $E^2(\gamma) \subset E^\infty(\gamma)$ for every comparison function γ, and*

$$\|f\|_{\infty,\gamma} \leq \|f\|_{2,\gamma}$$

for each $f \in E^2(\gamma)$.

(b) If, in addition, the sequence $\{n\gamma_n/\gamma_{n-1}\}$ is bounded, then $E^\infty(\gamma) \subset E^2(z^2\gamma(z))$, and there exists a constant $C < \infty$ such that

$$\|f\|_{2,z^2\gamma(z)} \leq C\|f\|_{\infty,\gamma}.$$

REMARKS: Part (a) of the Proposition justifies the statement made earlier that convergence in $E^2(\gamma)$ implies uniform convergence on compact subsets of the plane.

The statement of part (b) sacrifices some correctness for the sake of clarity. Strictly speaking, the function $z^2\gamma(z)$ is not a comparison function, because its first two Taylor coefficients are zero. So to be completely accurate we should have stated the theorem in terms of, for example, the function

$$\tilde{\gamma}(z) = a + bz + z^2\gamma(z), \tag{2}$$

where a and b are appropriately chosen positive numbers. However the important point is that the two functions have essentially the same growth as $|z| \to \infty$. Rather than obscure this idea with too much precision, we will allow ourselves such small inaccuracies whenever the occasion seems to demand them.

Finally, Proposition 1.4 implies that every element of $E^2(\gamma)$ has order and type no more than that of γ. In particular, if D is compact on $E^2(\gamma)$, then that space consists of entire functions of exponential type zero.

PROOF: (a) This part follows directly from the Cauchy-Schwarz inequality. For $f \in E^2(\gamma)$ and $z \in \mathrm{C}$ we have

$$\begin{aligned} |f(z)| &\leq \sum_{n=0}^{\infty} |\hat{f}(n)|\ |z|^n = \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|}{\gamma_n}\ \gamma_n |z|^n \\ &\leq \|f\|_{2,\gamma} \left(\sum_{n=0}^{\infty} \gamma_n^2 |z|^{2n} \right)^{1/2} \\ &\leq \|f\|_{2,\gamma}\ \gamma(|z|), \end{aligned}$$

where the "subadditivity of the power 1/2" has been used in the last line. The desired result follows immediately.

(b) This inequality requires a little more care. For each fixed $0 \leq r < \infty$, the sequence $\{\gamma_n r^n\}$ converges to zero, so it has a maximum value $\mu(r)$. This is the "maximum term" of the series $\sum \gamma_n r^n$. We claim that the hypothesis $\gamma_{n+1}/\gamma_n = O(1/n)$ insures that

$$\gamma(r) = O(r\mu(r)) \qquad (r \to \infty). \tag{3}$$

This is a standard result about entire functions of exponential type (see [**Val**, page 35] for more general inequalities, and [**Ros1**], [**Ros2**] for a beautiful probabilistic approach to the subject). To keep our exposition complete, we will include a proof, after first showing how the estimate yields the desired inequality.

So fix $f \in E^{\infty}(\gamma)$, and write $A = \|f\|_{\infty,\gamma}$. It is also convenient to adopt the notation

$$M_{\infty}(f, r) = \max_{|z|=r} |f(z)|$$

for $0 \le r < \infty$. Now fix a non-negative integer n and apply the corresponding Cauchy inequality to get

$$
\begin{aligned}
(4) \qquad |\hat{f}(n+1)| &\le M_\infty(f,r)r^{-(n+1)} \\
&\le A\gamma(r)r^{-(n+1)} \\
&\le CA\mu(r)r^{-n},
\end{aligned}
$$

where the last line follows from estimate (3) above, at least for $r \ge \gamma_0/\gamma_1$, and the constant C depends on this lower bound for r (the reason for this choice of lower bound will soon become apparent).

At this point the fact that the sequence $\{\gamma_n/\gamma_{n-1}\}$ decreases to zero becomes important, for it insures that

$$\mu(r) = \gamma_n r^n \quad \text{for} \quad \frac{\gamma_{n-1}}{\gamma_n} \le r \le \frac{\gamma_n}{\gamma_{n+1}},$$

where we define $\gamma_{-1} = 0$ to get the correct statement for $n = 0$. This simply asserts that each non-negative integer n gets to be the "maximum term index" for some r (possibly several n's may correspond to a single r). Thus for r as in the display above, (4) yields $|\hat{f}(n+1)| \le CA\gamma_n$. Since we are also assuming that $\gamma_n/\gamma_{n-1} = O(1/n)$, this last inequality insures that $f \in E^2(\tilde{\gamma})$, with the appropriate inequality holding between norms, where $\tilde{\gamma}$ is the comparison function defined by (2) above.

This proves part (b) of the Proposition, subject to verification of estimate (3) above. For this, observe that the condition $\gamma_n/\gamma_{n-1} = O(1/n)$ insures, via Stirling's formula, that for some constant $C < \infty$, independent of n,

$$\gamma_n \le \frac{C}{n!} \le C\left(\frac{e}{n}\right)^n.$$

Now fix $0 \le r < \infty$, and let N denote the unique integer for which

$$2er < N \le 2er + 1.$$

Then

$$\sum_{n=N}^{\infty} \gamma_n r^n \le C \sum_{n=N}^{\infty} \left(\frac{er}{n}\right)^n < C \sum_{n=N}^{\infty} 2^{-n} \le C,$$

so

$$\gamma(r) \le C + \sum_{n=0}^{N-1} \gamma_n r^n \le C + N\mu(r) \le C + (2er+1)\mu(r),$$

from which (3) follows, since $\mu(r) \to \infty$. ∎

2. Hypercyclicity

From now on we consider only admissible comparison functions γ. We remind the reader that by Proposition 1.1 and Corollary 1.2, the operators of differentiation and translation are therefore bounded on the Hilbert space $E^2(\gamma)$. The following result is the goal of this section.

2.1. Theorem. *For each admissible comparison function γ, the translation operator T_a is hypercyclic on $E^2(\gamma)$ for every $0 \ne a \in \mathbb{C}$.*

The full strength of admissibility is not going to be used in the proof of this result. In fact the result itself yields something a lot more general.

Corollary. *Suppose $a \ne 0$ and X is any Fréchet space of entire functions with the following properties:*

(1) *X contains the holomorphic polynomials as a dense subset.*
(2) *The topology of X is stronger than the topology of uniform convergence on compact subsets of the plane.*
(3) *T_a is continuous on X.*
(4) *$E^2(\gamma) \subset X$ for some admissible comparison function γ.*

Then T_a is hypercyclic on X.

Proof of Corollary: By Theorem 2.1 there exists $f \in E^2(\gamma)$ such that the orbit $\{T_a^n f\}$ is dense in $E^2(\gamma)$. By (2), and an elementary argument involving the

Closed Graph Theorem, the topology of $E^2(\gamma)$ is stronger than that of X, and this, along with (1) shows that the orbit of f is also dense in X. ∎

Upon taking $X = \mathcal{E}$ in this corollary, and recalling that for any comparison function there is an admissible one with smaller coefficients, we obtain Duyos-Ruiz's improvement of Birkhoff's theorem [**DuR**]: *If $0 \neq a$ and γ is a comparison function, then there is a hypercyclic vector for T_a on $\mathcal{E}$ that belongs to $E^2(\gamma)$ (or, equally well, because of Proposition 1.4, to $E^\infty(\gamma)$).*

We will break the proof of Theorem 2.1 into several parts. First of all, we need a sufficient condition for hypercyclicity.

2.2. PROPOSITION. *Let A be a bounded linear operator on a separable Banach space X. Suppose there exists a sequence $\{r_k\}$ of positive integers, strictly increasing to ∞, corresponding to which there is:*

(1) *A dense subset $X_0 \subset X$ such that $||A^{r_k}x|| \to 0$ for every $x \in X_0$, and*

(2) *A dense subset $Y_0 \subset X$ and a mapping $B : Y_0 \to Y_0$ such that $AB =$ identity on Y_0, and $||B^{r_k}y|| \to 0$ for every $y \in Y_0$.*

Then A is hypercyclic on X.

Separability of the underlying space is clearly a necessary condition for hypercyclicity. The first result of this type was discovered by Carol Kitai in her 1982 Toronto dissertation [**Kit**]. Kitai never published the theorem, and it was rediscovered a few years later by Gethner and Shapiro [**GeS**], who used it to unify the proofs of the theorems of Birkhoff, MacLane, and Rolewicz, and to discover hypercyclic behavior in many other settings. Since then the result has figured prominently in subsequent studies of hypercyclicity ([**BoS1**], [**Bos2**], [**GoS**], [**Her**], [**HeW**]). The version stated here is Corollary 1.4 of [**GoS**]. For completeness, we will provide a proof at the end of this section.

In order to appreciate the problems that arise in using this sufficient condition to prove Theorem 2.1, let us first sketch how it yields Birkhoff's theorem for the operator T_1 of "translation by one," acting on the space $\mathcal{E}$ of all entire functions.

Although not a Banach space, $\mathcal{E}$ is a Fréchet space, and that is all that is really needed for Proposition 2.2, if you interpret the "norm" of an element to be its distance to the origin. In the hypotheses of Proposition 2.2, take $A = T_1$, and $B = T_{-1}$ (the inverse of A in this case). Set

$$X_0 = \text{span } \{e^{\lambda z} : \text{Re } \lambda < 0\},$$

and

$$Y_0 = \text{span } \{e^{\lambda z} : \text{Re } \lambda > 0\}.$$

An elementary duality argument proves that X_0 and Y_0 are dense subspaces of $\mathcal{E}$ (see [GoS; section 5], for example). These spaces have been constructed specially so that $A^k \to 0$ pointwise on X_0, and $B^k \to 0$ pointwise on Y_0. Thus the hypothesis of Proposition 2.2 are satisfied, so $A = T_1$ is hypercyclic on $\mathcal{E}$. The case of translation by other non-zero complex numbers can either be done in similar fashion, or reduced easily to this special case.

This argument works as well for those Hilbert spaces, like $E^2(e^z)$, that contain exponential functions. However our interest here lies in much smaller spaces $E^2(\gamma)$; those for which the comparison function γ is of exponential type zero. In this case Proposition 1.4(a) shows that none of the exponentials $e^{\lambda z}$ used above to generate the crucial subspaces X_0 and Y_0 can belong to $E^2(\gamma)$, so we must construct these subspaces by different means. We will employ operator theory to reduce the problem to one in function theory, which we then solve by a classical construction.

To introduce the relevant operator theory, recall formula (1) occurring in the proof of Proposition 1.1, which represents the operator D of differentiation as a weighted backward shift relative to the orthonormal basis

$$e_n(z) = \gamma_n z^n \qquad (n = 0, 1, \dots)$$

for $E^2(\gamma)$. A little calculation shows that the adjoint operator D^* is, relative to the same basis, a weighted *forward* shift:

$$D^* e_n = \frac{(n+1)\gamma_{n+1}}{\gamma_n} e_{n+1} \qquad (n = 0, 1, \dots). \tag{1}$$

Since our viewpoint is that comparison functions become more interesting as they grow more slowly (i.e. as their Taylor coefficients γ_n tend to zero more rapidly), the most interesting comparison functions will satisfy the hypotheses of the next theorem, which is something of a landmark in the theory of weighted shifts. Its dual form will provide the approximation theorem that we seek.

To state these results, let A be a bounded operator on a Hilbert space $\mathcal{H}$, and suppose that relative to some orthonormal basis $\{e_n\}$, A has the representation

$$Ae_n = w_{n+1}e_{n+1} \qquad (n = 0, 1, \dots),$$

where $\{w_n\}$ is a bounded sequence of complex numbers. Then A is called a *weighted (*forward*) shift* with weights $\{w_n\}$ (relative to the basis $\{e_n\}$). A subset $\mathcal{S}$ of $\mathcal{H}$ is called *A-invariant if $A\mathcal{S} \subset \mathcal{S}$.*

2.3. The Unicellularity Theorem. *If $w_n \searrow 0$ and $\sum w_n^2 < \infty$, then the only closed, A - invariant subspaces of $\mathcal{H}$ are $\{0\}$ and the closed subspaces spanned by the basis elements*

$$\{e_n, e_{n+1}, \dots\} \qquad (n = 0, 1, \dots).$$

This result was originally proved by Donoghue [**Don**] for the weight sequence $w_n = 2^{-n}$. Later Nikol'skiĭ [**Nik**], and independently to Parrott and Shields (see [**Shl**, Cor. 1, page 105], proved the result under the weaker hypothesis of ℓ^p summability of the monotonic weight sequence for some $p < \infty$. Further results and references on this, and many other topics pertaining to weighted shifts can be found in Shields' survey article [**Shl**]. For completeness of exposition we will provide a proof of Theorem 2.3 at the end of this section, but we will say nothing more about the stronger "ℓ^p result."

Here is the "dual form" of Theorem 2.3.

2.4. Corollary. *Suppose $n\gamma_n/\gamma_{n-1} \searrow 0$, and*

$$\sum_{n=1}^{\infty} \left(\frac{n\gamma_n}{\gamma_{n-1}}\right)^2 < \infty.$$

Then every function in $E^2(\gamma)$ that is not a polynomial is a cyclic vector for D.

Proof: Suppose $f \in E^2(\gamma)$ is not cyclic for D. We must show that f is a polynomial. Let $\mathcal{M}$ denote the closed linear span of the successive derivatives of f. Since f is not cyclic, $\mathcal{M}$ is a closed, proper, D-invariant subspace of $E^2(\gamma)$, so its orthogonal complement $\mathcal{M}^\perp$ is a non-trivial D^*-invariant subspace. By Theorem 2.3, $\mathcal{M}^\perp$ contains the basis vectors $e_n, e_{n+1}, \ldots$ for some non-negative integer n. Since f is orthogonal to $\mathcal{M}^\perp$, it has to be a polynomial. ∎

Remarks: (a) In Corollary 2.4 the operator D can, of course, be replaced by any backward weighted shift with weights satisfying the hypotheses of Theorem 2.3. In this generality, the role of the "polynomials" is taken over by the (finite) linear combinations of basis vectors $\{e_n\}$. One can readily check that in this generality Corollary 2.4 is in fact *equivalent* to Theorem 2.3. Since every such linear combination is clearly non-cyclic for backward shifts, we see that the converse of Corollary 2.4 is also true.

(b) In view of the remarks following the statement of Theorem 2.3, the conclusion of Corollary 2.4 still holds if one only assumes p^{th} power summability in the hypothesis, for some $p < \infty$. We will see later (Remarks following Prop. 3.1) that this condition is equivalent to membership of D in the corresponding "Schatten p -class."

(c) Specialists have known versions of Corollary 2.4, in the context of various "small" Fréchet spaces of entire functions since at least the 1960's [**Tay**], and such results are periodically rediscovered (see [**Iyr**] and [**Grb**] for this result set in the space of entire functions of exponential type zero). In this regard, it is worth noting that the conclusion of Corollary 2.4 holds for any Fréchet space of entire functions that obeys the hypotheses of the Corollary of Theorem 2.1, and is the union of those spaces $E^2(\gamma)$ (γ as in Corollary 2.4) that it contains. For example, the reader can check that this is true of the space of entire functions of exponential type zero, and the space of entire functions of order zero.

With Corollary 2.4 in hand we can outline the strategy to be used in proving Theorem 2.1. First observe that we need only consider the operator T_1 of "translation by one." For once this special case has been established, the general version follows from the fact that the dilation operator

$$U_a : f(z) \mapsto f(az)$$

establishes a unitary equivalence between T_a acting on $E^2(\gamma)$ and T_1 acting on $E^2(\gamma(|a|z))$ (explicitly: $T_a = U_{1/a}T_1U_a$).

Next, by the argument of the Corollary to Theorem 2.1, it suffices to prove Theorem 2.1 for "small spaces," specifically the spaces $E^2(\gamma)$ where γ satisfies the hypotheses of Corollary 2.4.

So suppose γ satisfies the hypotheses of Corollary 2.4, and write $T = T_1$. We claim that in order to show T is hypercyclic on $E^2(\gamma)$, it suffices to find a *single* $f \in E^2(\gamma)$ such that $||T^{r_k}f||_{2,\gamma} \to 0$ for some sequence $\{r_k\}$ of positive integers that increases to ∞. Note that such a function f is necessarily transcendental. Once it has been found, Corollary 2.4 will guarantee that the subspace $X_0 = \text{span }\{D^n f\}_0^\infty$ is dense in $E^2(\gamma)$. Since each operator D^n is continuous on $E^2(\gamma)$ (Proposition 1.1), and commutes with T, it will follow that T^{r_k} tends to zero pointwise on X_0. Clearly the same will hold for the sequence of operators $\{(T_{-1})^{r_k}\}$ on the dense subspace $Y_0 = \text{span }\{D^n f(-z)\}_0^\infty$. Thus the hypotheses of Proposition 2.2 (the sufficient condition for hypercyclicity) will be satisfied with $A = T$ and $B = T_{-1}$, and this will establish that T is hypercyclic on $E^2(\gamma)$.

One final reduction: To produce the function f of the last paragraph, it is enough to show that for every comparison function γ there exists $f \in E^2(\gamma)$ such that $||T^{r_k}f||_{\infty,\gamma} \to 0$ for an appropriate sequence $\{r_k\}$. For once this has been established, then each time we are given γ, we will be able to find f and $\{r_k\}$ so that $||T^{r_k}f||_{\infty,\tilde{\gamma}} \to 0$, where $\tilde{\gamma}$ is a comparison function for which $\tilde{\gamma} = O(\gamma(r)/r^2)$. Then by Proposition 1.4(b), the function f will belong to $E^2(\gamma)$, and its translates will have the desired behavior in the norm of that space.

In summary, Theorem 2.1 will be proved in complete generality once we establish the following result.

2.5. PROPOSITION. *For each comparison function γ there exists a sequence $\{r_k\}$ of positive integers, strictly increasing to ∞, and a function $f \in E^2(\gamma)$ such that $||T^{r_k} f||_{\infty,\gamma} \to 0$ as $k \to \infty$.*

We are still a little way from being able to start the proof of this result. The function f we are going to produce will be an infinite product. We need a preliminary lemma to help us estimate the size of this product and its translates. Suppose for the moment that $\{r_k\}_1^\infty$ is any sequence of positive numbers, with

$$1 < r_1 < r_2 < r_3 < \cdots \nearrow \infty,$$

and α_k is a positive integer, to be regarded as the *multiplicity* of r_k. For $0 \le r < \infty$ let $n(r)$ be the number of points r_k that lie in the interval $[0, r]$, where r_k is counted with multiplicity α_k. More precisely, for each positive integer k,

$$n(r) = \alpha_1 + \alpha_2 + \cdots + \alpha_k \quad \text{for } r_k \le r < r_{k+1}. \tag{2}$$

We call $n(r)$ the *counting function* for the sequence $\{r_k\}$, with multiplicities $\{\alpha_k\}$. With these definitions, our fundamental growth estimate can be stated as follows.

2.6. LEMMA. *If $n(r) = O(\log r)$ as $r \to \infty$, then for some $\rho_0 > 0$,*

$$\prod_{k=1}^{\infty} \left(1 + \frac{r}{r_k}\right)^{\alpha_k} \le e^{2n(r)\log r}$$

for all $r \ge \rho_0$. Here the constant ρ_0 depends only on the sequences $\{r_k\}$ and $\{\alpha_k\}$.

PROOF: (cf. [**Boas**, section 3.5]). Let $H(r)$ denote the infinite product to be estimated. Then

$$\begin{aligned}
\log H(r) &= \sum_{k=1}^{\infty} \alpha_k \log\left(1 + \frac{r}{r_k}\right) \\
&= \int_1^\infty \log\left(1 + \frac{r}{t}\right)\, dn(t) \\
&= r \int_1^\infty \frac{n(t)\, dt}{t(t+r)}\,,
\end{aligned}$$

where the last line follows from an integration by parts, with the disappearance of the boundary terms aided by our growth hypothesis on the counting function. Note that the integral is allowed to start at 1 because the counting function vanishes on the interval $[0, r_1)$, and $r_1 > 1$. The hypothesis on $n(r)$ states that there exist positive constants ρ_0 and C such that

$$n(r) \leq C \log r \quad \text{for} \quad r \geq \rho_0. \tag{3}$$

Now fix $r > \rho_0$ and split the last integral into two parts:

$$\begin{aligned} I(r) &= \int_1^r \frac{n(t)\ dt}{t(t+r)} \\ &\leq n(r) \int_1^r \frac{dt}{t(t+r)} \\ &= \frac{n(r)}{r} \big[\log(1+r) - \log 2\big] \\ &< \frac{n(r)}{r} \log(1+r)\ , \end{aligned}$$

and

$$II(r) = \int_r^\infty \frac{n(t)\ dt}{t(t+r)} \leq C \int_r^\infty \frac{\log t\ dt}{t^2} = C \frac{1 + \log r}{r}\ ,$$

where the inequality is a result of (3) above. Thus for $r > \rho_0$,

$$\begin{aligned} \log H(r) &\leq r\big[I(r) + II(r)\big] \\ &< n(r) \log(1+r) + C(1 + \log r) \\ &< 2n(r) \log r, \end{aligned}$$

where, because $n(r) \nearrow \infty$, the last line holds upon suitable enlargement of the constant ρ_0. ∎

Proof of Proposition 2.5: Since the Taylor coefficients of γ are strictly positive, γ is not a polynomial, and $\gamma(r)$ is the maximum modulus of $\gamma(z)$ on the circle $|z| = r$. Thus

$$\frac{\gamma(r)}{r^k} \to \infty \qquad (r \to \infty) \tag{4}$$

for each positive integer k, so the numbers

$$\mu_k = \sup_{r \geq 1} (4r)^{2^{k+1}} \gamma(r)^{-\frac{1}{2}} \tag{5}$$

are all finite, and clearly they tend monotonically to ∞. We can therefore choose a sequence of integers $\{r_k\}$, each larger than 1, so that for any positive integer k,

$$r_{k+1} > 3r_k \ , \tag{6}$$

$$r_k > \sqrt{k\mu_k} \ , \tag{7}$$

$$r_k > e^{2^{k+1}} \ , \tag{8}$$

$$\log \gamma(r_k) > 16 \cdot 2^k \log r_k, \tag{9}$$

where (4) guarantees that the last choice is possible.

Let $n(r)$ be the counting function for the sequence $\{r_k\}$ just chosen, where the point r_k is assigned multiplicity 2^k. Thus by (2) above,

$$n(r) = 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 2 \qquad (r_k \leq r < r_{k+1}),$$

so inequality (8) implies that

$$n(r) < \log r \qquad (r \geq 1). \tag{10}$$

Similarly, inequality (9) provides the estimate that links $n(r)$ with $\gamma(r)$;it implies that

$$n(r) = 2^{k+1} - 2 < \frac{\log \gamma(r_k)}{8 \log r_k} \qquad (r_k \leq r < r_{k+1}).$$

Now the Hadamard Three Circles Theorem asserts that $\log M_\infty(\gamma, r)$, which in this case is just $\log \gamma(r)$, is a convex function of $\log r$. This implies that $\log \gamma(r)/\log r$, which is unbounded because γ is transcendental, is an increasing function of r for all sufficiently large r, say $r > \rho_1$. So, because $n(r)$ is constant on each interval $r_k \leq r < r_{k+1}$, the last inequality yields the estimate

$$n(r) < \frac{\log \gamma(r)}{8 \log r} \qquad (r > \rho_1). \tag{11}$$

At this point, let ρ be the largest of the numbers ρ_0, ρ_1, and 1, and note once and for all that ρ depends only on the comparison function γ.

With the estimates for $n(r)$ in hand, we can now construct the function f that is the goal of this proof. We claim that it is given by the infinite product

$$f(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{r_k}\right)^{2^k} .$$

Note that condition (6) (or even better, (8)) insures that $\sum 2^k/r_k < \infty$, so the convergence of this product to an entire function is not in doubt. The fact that this function belongs to $E^\infty(\gamma)$ will emerge as a byproduct of our estimates on its translates.

We begin these estimates by fixing a positive integer n, and a point $z \in \mathrm{C}$ with $|z| > \rho$.

For the purpose of estimating $||T^{r_n} f||_{\infty,\gamma}$ we consider the factorization $f = P_{n-1}Q_nR_{n+1}$, where

$$P_{n-1}(z) = \prod_{k=1}^{n-1} \left(1 - \frac{z}{r_k}\right)^{2^k} ,$$

$$Q_n(z) = \left(1 - \frac{z}{r_n}\right)^{2^n} ,$$

and

$$R_{n+1}(z) = \prod_{k=n+1}^{\infty} \left(1 - \frac{z}{r_k}\right)^{2^k} .$$

In dealing with these products, a little algebraic identity will prove very useful (cf. [**DuR**]):

$$1 - \frac{z+b}{a} = \left(1 - \frac{b}{a}\right)\left(1 - \frac{z}{a-b}\right) . \tag{12}$$

When applied to the tail end of the product for f, this identity yields

$$R_{n+1}(z + r_n) = R_{n+1}(r_n) \prod_{k=n+1}^{\infty} \left(1 - \frac{z}{r_k - r_n}\right)^{2^k} .$$

Now each term in the product for $R_{n+1}(r_n)$ is a positive number < 1, so the last equation yields, for every $z \in \mathbf{C}$,

$$|R_{n+1}(z + r_n)| \le \prod_{k=n+1}^{\infty} \left(1 + \frac{|z|}{r_k - r_n}\right)^{2^k} .$$

According to condition (6), for $k \ge n+1$ we have

$$r_k - r_n > r_k - r_{k-1} > r_{k-1},$$

so the inequality above yields

$$|R_{n+1}(z + r_n)| < \prod_{k=n+1}^{\infty} \left(1 + \frac{|z|}{r_{k-1}}\right)^{2^k} \le \prod_{k=1}^{\infty} \left(1 + \frac{|z|}{r_k}\right)^{2^{k+1}} .$$

The product on the right is formed from the sequence $\{r_k\}$ where the point r_k has multiplicity 2^{k+1} (twice its original multiplicity). The counting function for $\{r_k\}$ with these new multiplicities is $2n(r)$, which according to condition (10) satisfies the hypothesis of Lemma 2.6. This Lemma, along with the last inequality, yield

$$|R_{n+1}(z + r_n)| < e^{4n(|z|)\log|z|} < \gamma(|z|)^{\frac{1}{2}} \tag{13}$$

where the final inequality follows from (11).

Having satisfactorily estimated the tail of the infinite product for $f(z + r_n)$, we concentrate attention on the front end, and again use identity (12) to obtain:

$$\begin{aligned} |P_{n-1}(z + r_n)| &= |P_{n-1}(r_n)| \prod_{k=1}^{n-1} \left|1 - \frac{z}{r_k - r_n}\right|^{2^k} \\ &\le \prod_{k=1}^{n-1} \left[(1 + r_n)(1 + |z|)\right]^{2^k} \\ &\le (4r_n|z|)^{2+4+\cdots+2^{n-1}} \\ &= (4r_n|z|)^{2^n - 2}, \end{aligned}$$

where we have used in the second line the fact that $r_n - r_k > 1$ (as in the previous case, from (6)), and in the third line the fact that both r_n and $|z|$ exceed 1. Thus

$$\begin{aligned}
&|P_{n-1}(z+r_n)|\ |Q_n(z+r_n)| \\
&\qquad\qquad \leq (4r_n|z|)^{2^n-2}\left|\frac{z}{r_n}\right|^{2^n} \\
&\qquad\qquad < \frac{(4|z|)^{2^{n+1}}}{r_n^2} \\
&\qquad\qquad < \frac{\mu_n\gamma(|z|)^{\frac{1}{2}}}{r_n^2} \\
&\qquad\qquad < \frac{1}{n}\gamma(|z|)^{\frac{1}{2}},
\end{aligned}$$

where the last two inequalities follow respectively from (5) (the definition of μ_n) and condition (7) that was placed on the sequence $\{r_k\}$. Thus (13) and the last inequality yield

$$|f(z+r_n)| = |P_{n-1}Q_nR_{n+1}(z+r_n)| \leq \frac{1}{n}\ \gamma(|z|) \tag{14}$$

whenever $|z| > \rho$.

For $|z| \leq \rho$ we use (14) (with $|z| = \rho$) along with the maximum principle and the fact that $\gamma(r)$ is strictly positive and increasing on $[0, \infty]$ to get:

$$|f(z+r_n)| \leq \max_{|\zeta|=\rho} |f(\zeta + r_n)| \leq \frac{1}{n}\gamma(\rho) < \frac{1}{n}\frac{\gamma(\rho)}{\gamma(0)}\gamma(|z|).$$

This inequality, along with (14) gives the desired result:

$$||T^{r_n}f||_{\infty,\gamma} \leq \frac{1}{n}\frac{\gamma(\rho)}{\gamma(0)} \to 0$$

as $n \to \infty$. This completes the proof of Proposition 2.5, and with it, that of Theorem 2.1. ∎

Appendix: Two proofs. We close this section with the proofs that were promised for Proposition 2.2 and Theorem 2.3. We emphasize that these proofs are presented for purely expository reasons, and no originality is being claimed.

Proof of Proposition 2.2: Recalling that the Banach space X is separable, let $\{\mathcal{O}_\nu\}$ be an enumeration of its open balls that have rational radii and centers in some fixed countable dense set. The collection of hypercyclic vectors for A (which we want to show is non-empty) can then be written as

$$\bigcap_\nu \bigcup_n A^{-n}(\mathcal{O}_\nu)$$

where $\cup_n A^{-n}(\mathcal{O}_\nu)$ is an open set, because the operator A is continuous. We claim that each of these open sets is dense in X. By the Baire Category Theorem this will yield the desired non-emptyness of the set of hypercyclic vectors for A (in fact, it will show this set to be a dense G_δ).

To this end, fix a non-empty open subset $\mathcal{O}$ of X. We have to show that for each index ν, the intersection of $\cup_n A^{-n}(\mathcal{O}_\nu)$ with $\mathcal{O}$ is non-empty. So fix the index ν, and note that by the assumed density of the sets X_0 and Y_0 there exist points $x_0 \in X_0 \cap \mathcal{O}$ and $y_0 \in Y_0 \cap \mathcal{O}_\nu$. For simplicity, write A_k for A^{r_k}, and define B_k similarly. Then by the definition of Y_0 we have $B_k y_0 \to 0$, so upon writing

$$x_k = x_0 + B_k y_0$$

we obtain from the linearity of A, the fact that AB = identity on Y_0, and the definition of X_0:

$$A_k x_k = A_k x_0 + y_0 \to y_0 \qquad (k \to \infty).$$

Thus for k sufficiently large, $x_k \in \mathcal{O}$ and $A_k x_k \in \mathcal{O}_\nu$, hence $\mathcal{O} \cap A^{-r_k}(\mathcal{O}_\nu) \neq \emptyset$, as desired. ∎

Proof of Theorem 2.3: The argument is best split into two steps, the first of which contains most of the work.

Special Case. If $f \in \mathcal{H}$ and $< f, e_0 > \neq 0$, then f is cyclic for A.

To see why this is so, write $f = \sum a_n z_n$, where $a_0 \neq 0$. Without loss of generality we may assume $a_0 = 1$. Write $f_0 = e_0$, and for $n = 1, 2, \ldots$ set

$$(1) \qquad f_n = \frac{A^n f}{w_1 w_2 \cdots w_n} = e_n + \sum_{j=1}^{\infty} \frac{w_{j+1} w_{j+2} \cdots w_{j+n}}{w_1 w_2 \cdots w_n} \; a_j e_{j+n} \; .$$

Now recall that the weight sequence $\{w_n\}$ decreases monotonically, so

$$\text{(2)} \qquad \frac{w_{j+1}w_{j+2}\cdots w_{j+n}}{w_1 w_2 \cdots w_n} \leq \frac{w_2 w_3 \cdots w_{n+1}}{w_1 w_2 \cdots w_n} = \frac{w_{n+1}}{w_n} \ .$$

Thus

$$\text{(3)} \qquad \sum_{n=1}^{\infty} \|f_n - e_n\|^2 \leq \frac{\|f\|^2}{w_1^2} \sum_{2}^{\infty} w_n^2 < \infty \ .$$

A well-known theorem of Paley and Wiener [**PW**, page 100] asserts that the sequence $\{f_n\}$, which (3) exhibits as asymptotically just a small perturbation of an orthonormal basis, is in fact a *Schauder basis* for $\mathcal{H}$, so in particular its closed linear span is dense (see also [**You**, Theorem 10, page 38], [**Woj**, II.B, Prop 15], and [**Hal**, Problem 12]). Thus f is a cyclic vector for A.

To understand the idea behind the Paley-Wiener theorem, suppose that the sum on the left side of (3) is < 1. Then a simple argument shows that the operator S defined initially from span $\{e_n\}$ to $\mathcal{H}$ by $Se_n = f_n$ $(n \geq 0)$ extends to a bounded operator on $\mathcal{H}$ with $\|S - I\| < 1$. Thus S is invertible on $\mathcal{H}$, so the sequence $\{f_n\}$ inherits the basis property of $\{e_n\}$. In case this sum is ≥ 1, one applies this result to the linear span of the vectors $\{e_n, e_{n+1}, \cdots\}$ for n sufficiently large, and deals separately with the remaining subspace of dimension n.

The general case. Suppose $\mathcal{M} \neq \{0\}$ is a closed subspace of $\mathcal{H}$ that is invariant for A. Then the formula

$$\nu = \inf\{n :< f, e_n > \neq 0, f \in \mathcal{M}\}$$

defines a non-negative integer for which $\mathcal{M} \subset \mathcal{H}_\nu$. We will be finished if we can show that $\mathcal{M} = \mathcal{H}_\nu$.

Now there exists $f \in \mathcal{M}$ with $< f, e_\nu >= 0$. Apply the *Special Case* with $\mathcal{H}$ replaced by $\mathcal{H}_\nu$ and A by its restriction to $\mathcal{H}_\nu$. *Conclusion:* The vectors $\{f, Af, A^2 f, \ldots\}$, all of which belong to $\mathcal{M}$, span a dense subspace of $\mathcal{H}_\nu$. Thus $\mathcal{M} = \mathcal{H}_\nu$. ∎

3. Compact Perturbations of the Identity

Having proved the hypercyclicity of translation operators on "admissible" spaces $E^2(\gamma)$, we now examine the compactness of their differences with the identity.

We will show that by controlling the rate at which the sequence $\{n\gamma_n/\gamma_{n-1}\}$ tends to zero (i.e., by controlling the size of the space $E^2(\gamma)$), we can make the difference $T_a - I$ "as compact as desired." Our first goal is to describe precisely what this means.

How to measure compactness. Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$. For n a non-negative integer, the n-th *approximation number* of A, denoted by $\alpha_n(A)$, is defined to be the distance, measured in the operator norm, from A to the closed subspace of operators on $\mathcal{H}$ of rank $\leq n$. More precisely, if $\mathcal{L}(\mathcal{H})$ denotes the collection of bounded linear operators on $\mathcal{H}$, then

$$\alpha_n(A) = \inf\{||A - F|| : F \in \mathcal{L}(\mathcal{H}), \text{rank } F \leq n\}.$$

According to our definition, $\alpha_0 = ||A||$, and the sequence of approximation numbers is monotone decreasing. Its limit is zero if and only if A is compact; this just restates the fact that an operator is compact if and only if it is a norm-limit of finite rank operators. Here is an extreme case: the sequence of approximation numbers is eventually zero if and only if the operator is of finite rank. Thus it makes sense to regard one operator as being "more compact" than another if its sequence of approximation numbers tends more rapidly to zero.

After the finite rank operators, the two best known sub-classes of compact operators are the *Hilbert-Schmidt class* ($\sum \alpha_n(A)^2 < \infty$), and the *trace class* ($\sum \alpha_n(A) < \infty$). More generally, if $0 < p < \infty$ and $\sum \alpha_n(A)^p < \infty$ then we say A belongs to the *Schatten p-class*. According to the view being advanced here, as p decreases, the compactness of the operators in the Schatten p-class increases.

Approximation numbers of the differentiation operator. As an instructive example that will be important later on, let us estimate the approximation numbers for the differentiation operator on $E^2(\gamma)$.

3.1. Proposition. *If γ is admissible, then*

$$\alpha_n(D) \le \frac{(n+1)\gamma_{n+1}}{\gamma_n}$$

for any non-negative integer n.

Proof: Recall that "γ admissible" means that the sequence $w_n = n\gamma_n/\gamma_{n-1}$ is monotonically decreasing. From last line of the proof of Proposition 1.1, we see that $||D - D_n|| \le w_{n+1}$ where D_n is an operator of rank n. Thus $\alpha_n(D) \le w_{n+1}$ as desired. ∎

This result shows, for example, that D is in the Schatten p-class on $E^2(\gamma)$ whenever

$$\sum_1^\infty \left(\frac{n\gamma_n}{\gamma_{n-1}}\right)^p < \infty.$$

Note that for $p = 2$ this condition is precisely the hypothesis of Corollary 2.4, the crucial approximation result of the last section.

In fact, more is true: *there is equality in the statement of Proposition 3.1*, so the condition above is also *necessary* for D to be in the Schatten p-class. We do not emphasize this fact since it is not required for the sequel. Nevertheless the full story is this: if A is a compact operator on Hilbert space, then the sequence of eigenvalues of the positive compact operator A^*A, when arranged in decreasing order, coincides with the sequence squares of approximation numbers of A (see [**DS**, Part II, Chapter XI, section 9], or [**Woj**, sec. III.G] for the details). Recall from section 2 that the backward shift representation of D gives rise to a corresponding forward shift representation for the adjoint operator D^*. Taken together, these representations show that relative to the orthonormal basis $\{e_n\}$, the operator D^*D has the diagonal matrix $\{0, w_1^2, w_2^2, \dots\}$, where w_n is the weight given by (1) above. Thus $\alpha_n(D) = w_{n+1}$ for each non-negative integer n.

Approximation numbers of $T_a - I$. This brings us to the main result of this section, most of whose proof asserts that the approximation numbers of $T_a - I$ decrease to zero at the same rate as those of D.

THEOREM 3.2. *Suppose $\{\omega_n\}$ is a sequence of positive numbers that tends monotonically to zero, and $\epsilon > 0$ is given. Then there exists a comparison function γ and a positive number δ such that on $E^2(\gamma)$ the operator T_a is hypercyclic for each $a \neq 0$,*

$$\alpha_n(T_a - I) = o(\omega_n) \qquad \text{as } n \to \infty,$$

and $||T_a - I|| < \epsilon$ for all $|a| < \delta$.

PROOF: Define the sequence $\{\gamma_n\}$ by: $\gamma_0 = \omega_0$ and for n positive,

$$\gamma_{n+1} = \frac{\omega_n^2 \gamma_n}{n+1}.$$

Then

$$\frac{(n+1)\gamma_{n+1}}{\gamma_n} = \omega_n^2,$$

so $\gamma(z) = \sum \gamma_n z^n$ is a comparison function such that T_a $(a \neq 0)$ is hypercyclic on $E^2(\gamma)$ (Theorem 2.1), and by Proposition 3.1, the differentiation operator D has approximation numbers $\alpha_n(D) = w_n^2 = o(w_n)$.

Now the function

$$\Psi(z) = \frac{e^{az} - 1}{az} = \sum_{k=0}^{\infty} \frac{(az)^n}{(n+1)!}$$

is entire, so by Proposition 1.3 the series

$$\Psi(D) = \sum_{k=0}^{\infty} \frac{(aD)^n}{(n+1)!} \tag{2}$$

defines a bounded operator on $E^2(\gamma)$. By Corollary 1.2,

$$T_a - I = e^{aD} - I = aD\Psi(D).$$

Thus

$$||T_a - I|| < |a| \; ||D\Psi(D)|| < \epsilon$$

as long as $|a| < \epsilon / ||D\Psi(D)||$, which proves the statement about operator norms.

The assertion about approximation numbers follows from the easily verified fact that if A and B are bounded operators on Hilbert space, then for each n,

$$\alpha_n(AB) \leq ||B||\alpha_n(A).$$

This, along with (2) above shows that

$$\alpha_n(T_a - I) \leq |a| \; ||\Psi(D)|| \; \alpha_n(D) = o(\omega_n),$$

as desired. ∎

This result, along with Theorem 2.1, accomplishes the operator theoretic objective of this paper, which for completeness we state as the next result.

COROLLARY. *Suppose $\mathcal{H}$ is a Hilbert space, $\{\omega_n\}$ is a sequence of positive numbers that tends monotonically to zero, and $\epsilon > 0$ is given. Then there exists a compact operator K on $\mathcal{H}$ such that: $||K|| < \epsilon$, $\alpha_n(K) = o(\omega_n)$, and $I + K$ is hypercyclic on $\mathcal{H}$.*

4. CONCLUDING REMARKS AND OPEN PROBLEMS

Hypercyclicity vs. finite rank. Because of the following result of Kitai [**Kit**, Cor. 2.4], these two concepts are "mutually orthogonal."

4.1. PROPOSITION. *Suppose T is a bounded linear operator on Hilbert space. If the adjoint of T has an eigenvalue, then T is not hypercyclic.*

PROOF: According to the hypothesis, there is a non-zero vector $y \in \mathcal{H}$, and a complex number λ for which $T^*y = \lambda y$. Now if $x \in \mathcal{H}$ is hypercyclic for T, then the collection of complex numbers $\{< T^n x, y >\}_0^\infty$ will be dense in the plane. But for each non-negative integer n,

$$< T^n x, y >=< x, T^{*n} y >=< x, \lambda^n y >= \bar{\lambda}^n < x, y >,$$

and one easily checks that the set of complex numbers defined by the right side of this equation, as n ranges through the non-negative integers, is not dense in the plane. ∎

Though we prefer stay within the friendly confines of Hilbert space, this result, as well as those immediate consequences discussed here, hold for Banach spaces, and with suitably modified proofs.

To get some idea of the utility of Proposition 4.1, observe how readily it shows that *no finite dimensional Hilbert space supports a hypercyclic operator* [**Kit**, Theorem 1.2]. (*Proof.* The dual space is also finite dimensional, so the adjoint of any operator on the original space has an eigenvalue.) Similarly, no finite rank operator can be hypercyclic, and with a little more work one can show that the same is true of compact operators. Thus our standard assumption that γ is an admissible comparison function of exponential type zero (i.e., $n\gamma_n/\gamma_{n-1} \searrow 0$) implies that the operator D is not hypercyclic on the space $E^2(\gamma)$, simply because it is compact on that space.

The result below, which was mentioned in the Introduction, belongs to the same circle of ideas.

4.2. Corollary. *On Hilbert space, no perturbation of the identity by a finite rank operator is hypercyclic.*

Proof: Let F be a finite rank operator on $\mathcal{H}$, and write $T = I + F$. Then $T^* = I+F^*$, and F^* is also a finite rank operator. By our remarks about finite dimensional spaces, we may without loss of generality assume that $\mathcal{H}$ is infinite dimensional. Thus F^* has a non-trivial null space, so 1 is an eigenvalue of T^*. By Proposition 4.1, the operator T is therefore not hypercyclic. ∎

Paul Bourdon has obtained a variant of Proposition 4.1 (unpublished): *If the adjoint of a bounded operator has a bounded orbit, then the original operator is not hypercyclic.* Bourdon has also shown that no finite dimensional *real* Banach space supports a hypercyclic operator. The proof is more difficult than for the complex case, since eigenvalues are not so readily available.

Hypercyclicity and spectra. It follows from Propositon 1.3 that the operators $T_a - I$ that figure in Theorem 3.2 are, in addition to being compact, also *quasinilpotent*. The next result shows that this is no accident.

4.3. PROPOSITION. *If K is a compact operator on Hilbert space, and $I+K$ is hypercyclic, then K is quasinilpotent.*

PROOF: The adjoint operator K^* is also compact, so by the Riesz theory, if it had any non-zero spectral points, these would be eigenvalues. But this cannot happen, since eigenvalues of K^* would give rise to eigenvalues of $(I+K)^*$, and by by Proposition 4.1, this would contradict the hypercyclicity of $I+K$. Thus only the point zero belongs to the spectrum of K, so $\|K^n\|^{1/n} \to 0$ by the spectral radius formula. ∎

Now Kitai [**Kit**, Theorem 2.8] has shown that if an operator is hypercyclic, then its spectrum must intersect the unit circle (Kitai actually proves that every *component* of the spectrum must intersect the circle). Thus *hypercyclic perturbations of the identity by compacts have the minimal spectrum allowed a hypercyclic operator: a single point on the unit circle.*

Hypercyclicity for other differential operators? We return to $E^2(\gamma)$, under the assumption that γ is an admissible comparison function of exponential type zero. By Propositions 1.2 and 1.3, the operator D is therefore compact and quasinilpotent on $E^2(\gamma)$. Moreover, if Φ is a function that is holomorphic in a neighborhood of zero, then Proposition 1.3 (for the case $\tau = 0$) guarantees that the operator $\Phi(D)$ obtained by substituting D for z in the power series expansion of Φ is bounded on $E^2(\gamma)$. Now $\Phi(z) = \Phi(0) + z\Psi(z)$ where Ψ is holomorphic in a neighborhood of zero, so

$$\Phi(D) = \Phi(0) + D\Psi(D)$$

where $\Psi(D)$ is a bounded operator on $E^2(\gamma)$. Since D is quasinilpotent, so is $D\Psi(D)$, hence the spectrum of $\Phi(D)$ is just the singleton $\{\Phi(0)\}$.

By the result of Kitai mentioned above, $\Phi(D)$ has a chance to be hypercyclic only if $|\Phi(0)| = 1$. Section 2 of this paper has been devoted to proving hypercyclicity in the special case $\Phi(z) = e^{az}$. What about all the other cases? Nothing seems to be known; we do not know, for example if the operator $I + D$ is hypercyclic on $E^2(\gamma)$.

Large Hilbert spaces. In the Introduction we commented that large Hilbert spaces $E^2(\gamma)$ pose fewer obstacles to the hypercyclicity of translations than do small ones. By way of explanation, consider the comparison function $\gamma(z) = e^z$. Then $n\gamma_n/\gamma_{n-1} = 1$, so translation is bounded on $E^2(e^z)$, and more generally, by Proposition 1.3, $\Phi(D)$ is a bounded operator on $E^2(e^z)$ whenever Φ is holomorphic in a neighborhood of the closed unit disc $\bar{U}$. In this case the differentiation operator is most assuredly *not* compact on $E^2(e^z)$, since (for example) every point λ of the open disc U is an eigenvalue (for the eigenvector $e^{\lambda z}$).

In this situation it is easy to modify our proof of Birkhoff's theorem to give the following result, which is reminiscent of the adjoint multiplier theorem of Godefroy and Shapiro that was mentioned in the Introduction: *If $\Phi(z)$ is holomorphic in a neighborhood of the closed unit disc, and $\Phi(U)$ intersects the unit circle, then the operator $\Phi(D)$ is hypercyclic on $E^2(e^z)$.*

Thus, for example, every scalar multiple of T_a $(a \neq 0)$ is hypercyclic on $E^2(e^z)$, whereas the remarks above show that if g is of exponential type zero, then only multiples of T_a by *unimodular* constants can be hypercyclic (the proof of Theorem 2.1 shows that all such multiples actually *are* hypercyclic).

By a slightly modified argument we can also show that D itself is hypercyclic on $E^2(e^z)$ (in Proposition 2.2 take X_0 to be the span of the exponentials $e^{\lambda z}$ where $|\lambda| < 1$, but take Y_0 to be the collection of polynomials). This raises the following question: *If the Φ-image of the closed unit disc intersects the unit circle, is $\Phi(D)$ hypercyclic on $E^2(e^z)$?* For example, we do not even know if the operator $(I + D)/2$ is hypercyclic on this space.

Paley-Wiener spaces. If translation is bounded on some "reasonable" Hilbert space of entire functions, then must it be hypercyclic? As shown by the following example, the answer is "no."

For $\tau > 0$ let W^2_τ denote the space of entire functions f for which the collection of horizontal L^2 means

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx$$

is bounded for all real numbers y. This is the *Paley-Weiner space*, and it is well known that the restrictions of its elements to the real line form a closed subspace of $L^2(\mathrm{R})$. Thus W^2_τ is a space of entire functions that can be regarded as a closed subspace of the Hilbert space $L^2(\mathrm{R})$ (the key to this result is the *Paley-Wiener theorem*, which asserts that the complex Fourier transform establishes an isometric isomorphism of $L^2([-\tau,\tau])$ onto W^2_τ [**Rud2**, Ch. 19]).

Once this is established, it becomes clear that translation operators T_a for a real act boundedly on W^2_τ. In fact, they are all *isometries*, so none is hypercyclic!

General hypercyclic operators. Some very basic questions about the general notion of hypercyclicity remain to be answered. For example, Kitai asks if the square of a hypercyclic operator is hypercyclic [**Kit**, Ch. 2, page 2-9], and Herrero (personal communication) posed the same question about the direct sum of a hypercyclic operator with itself.

If an operator A satisfies the hypotheses of our sufficient condition for hypercyclicity (Proposition 2.2), *with the sequence $\{r_k\}$ equal to all the positive integers*, then it is easy to check that its square and its direct sum with itself also satisfy these hypotheses, so both these operators are hypercyclic. Most of the operators previously considered in the literature satisfy these stronger hypotheses.

In the other direction, Hector Salas [**Sal**] has shown that the direct sum of two (different) hypercyclic operators need not even be cyclic. This construction has also been described in [**Her**].

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