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MACKEY TOPOLOGIES, REPRODUCING KERNELS, AND DIAGONAL MAPS ON THE HARDY AND BERGMAN SPACES

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1. Introduction. Let U denote the open unit disc in the complex plane, T the unit circle; and $d\lambda$ and dm Lebesque measure on U and T respectively, both normalized to have total mass 1. The *Hardy space* H^p consists of all functions f analytic in U for which

$$\|f\|_p^p = \sup_{0 \le r < 1} \int_T |f(r\omega)|^p \, dm(\omega) < \infty,$$

and the *weighted Bergman space* A_α^p $(\alpha > -1)$ consists of those f analytic on U for which

$$\|f\|_{p,\alpha}^p = \int_U |f(z)|^p (1 - |z|)^\alpha \, d\lambda(z) < \infty.$$

If $1 \le p < \infty$ these are Banach spaces in the obvious norms. In this paper, however, we are interested in the range $0 < p < 1$, in which case the appropriate metrics are

$$d(f, g) = \|f - g\|_p^p \quad \text{for} \quad H^p$$

and

$$d(f, g) = \|f - g\|_{p,\alpha}^p \quad \text{for} \quad A_\alpha^p.$$

These metrics turn H^p and A_α^p respectively into *F-spaces* (complete, metrizable linear topological spaces) which are *not* locally convex, but nevertheless have enough continuous linear functionals to separate points (the evaluation functionals $f \to f(z)$ for $z \in U$, for example: see [1, page 37 and sec. 7.4] and [8] for H^p, [11] for A_α^p, and also section 2 of this paper).

In 1932 Hardy and Littlewood showed that $H^p \subset A_{1/p-2}^1$ for $0 < p < 1$, the inclusion map being continuous [5; Theorem 31, pp. 411–412] (reassurance: in this paper $a/b - c$ always means $(a/b) - c$). Recently Duren, Romberg, and Shields [2; Theorems 1 and 7] explicitly determined the dual spaces of H^p and $A_{1/p-2}^1$; and found them to be *the same* in the sense that every continuous linear functional on $A_{1/p-2}^1$ restricts to one on H^p, and every continuous linear functional on H^p extends uniquely to one on $A_{1/p-2}^1$. In short, the restriction

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map $\lambda \to \lambda|_{H^p}$ takes the dual of $A_{1/p-2}{}^1$ onto that of H^p. Using standard results about linear topological spaces (see section 2) this result can be rephrased as follows: *for* $0 < p < 1$ *the norm of* $A_{1/p-2}{}^1$ *induces the* Mackey topology (*strongest locally convex topology yielding the same dual*) *on* H^p ; *and* $A_{1/p-2}{}^1$ *can be regarded as the Mackey-completion of* H^p.

In general the Mackey topology of a non locally convex F-space with separating dual is metrizable and strictly weaker than the original topology [12, Prop. 3, page 641]. Since it plays a crucial role in the duality theory of these spaces (see [12, proof of Theorem 1] or [13] for example), we would like to be able to compute the Mackey topology as explicitly as possible in concrete situations. The usual method is the one just described for H^p : find a Fréchet space (locally convex F-space) which contains the original one and has the same dual in the sense of the last paragraph. The Mackey topology of the original space then turns out to be the restriction of the topology of the containing space (see section 2 for more details). Unfortunately this method requires explicit calculation of the duals of both the original and containing spaces.

In this paper we advocate a method for computing the Mackey topology of a non locally convex F-space without determining any dual spaces. When applied to H^p it yields a new proof of the theorem of Duren, Romberg, and Shields; when applied to $A_\alpha{}^p$ it shows that the Mackey topology is induced by the norm $||\cdot||_{1,\sigma}$ where $\sigma = (\alpha + 2)/p - 2$ $(0 < p < 1)$. In addition there are surprising connections with the behavior of the diagonal map on the H^p spaces of a polydisc.

The core of our method is the following result, which we prove in the next section.

THEOREM 1. *Suppose E is an F-space whose dual separates points. Then the Mackey topology of E is the unique locally convex topology τ on E such that*

(1) *τ is weaker than the E-topology, and*

(2) *the E-closure of the absolutely convex hull of each E-neighborhood of zero contains a τ-neighborhood of zero.*

By the *absolutely convex hull* of a set S we mean the collection of finite sums $\sum \lambda_i s_i$ where $s_i \in S$ and the λ_i are scalars with $\sum |\lambda_i| \le 1$. Since every linear topological space has a local base consisting of circled sets [7; Theorem 5.2, page 35] we could as well have used the more usual notion of "convex hull" in the statement of Theorem 1. However as stated the theorem is easier to apply.

To see how Theorem 1 works in a simple case, let us find the Mackey topology of the sequence space l^p $(0 < p < 1)$, using first the "old method", and then the Theorem. Both methods require the elementary fact that $l^p \subset l^1$, and the l^1 topology is weaker on l^p than the original topology. This is just condition (1) of Theorem 1 with $E = l^p$ and $\tau =$ the l_1 topology. It also shows that every continuous linear functional on l^1 restricts to one on l^p. Conversely, each continuous linear functional on l^p can be represented in the usual way by a bounded sequence [7, sec. 14, Problem M, Page 130], so l^p and l^1 have the same dual in the sense described previously (namely l^∞), hence l^1 induces the Mackey topology on l^p.

Theorem 1 provides a more direct approach. Let $E = l^p$ and $\tau =$ the l^1 topology on l^p. We have already pointed out that condition (1) of the Theorem holds, so we need only check condition (2). Now each x in l^p whose l^1 norm is ≤ 1 can be written as

$$x = \sum_{n=1}^{\infty} \xi_n e_n \tag{1.1}$$

where e_n is the nth standard basis vector (1 in the nth place and 0 elsewhere), and (ξ_n) is a scalar sequence with $\sum |\xi_n| < 1$. Thus the partial sums of the series in (1.1) are absolutely convex combinations of the vectors $(e_n)_1^\infty$. Moreover the vectors e_n lie in the l^p-unit ball, and the partial sums converge to x in l^p. This shows that the l^p closure of the absolutely convex hull of the l^p unit ball contains a τ-neighborhood of zero: those members of l^p with l^1 norms ≤ 1. Thus (after some scalar multiplication) we see that condition (2) is satisfied, which completes the "dualless" proof that l^1 induces the Mackey topology on l^p $(0 < p < 1)$.

The strategy for H^p and $A_\alpha{}^p$ $(0 < p < 1)$ is similar: we take $E = H^p$ or $A_\alpha{}^p$ and let τ be the topology induced by the norm $\|\cdot\|_{1,\sigma}$ where $\sigma = 1/p - 2$ for H^p and $(\alpha + 2)/p - 2$ for $A_\alpha{}^p$. Condition 1 of Theorem 1 follows from the previously mentioned result of Hardy and Littlewood (immediately for H^p, after a little work for $A_\alpha{}^p$). Condition (2) is verified, not by using a basis as was done for l^p, but by using a *reproducing kernel* to represent elements of H^p or $A_\alpha{}^p$ as limits of appropriate absolutely convex combinations. The details of this occupy sections 3 and 4, with section 2 reserved for background material and the proof of Theorem 1.

In the final section we connect these results with an open problem about the diagonal map on the H^p spaces of a polydisc.

2. Background material and proof of Theorem 1. If E is a real or complex linear space and F is a subspace of the algebraic dual of E, then there is a unique strongest locally convex topology μ on E for which F is precisely the space of μ-continuous linear functionals [7; sec. 18, page 173]. μ is called the *Mackey topology* of the pair (E, F), and denoted by $m(E, F)$: it is Hausdorff if and only if F separates points of E.

If E is a linear topological space with (topological) dual E', and the topology of E is already $m(E, E')$, then we call E a *Mackey space*. Theorem 1 is an immediate consequence of the following basic result [7; Corollary 22.3, page 210]:

PROPOSITION. *Every pseudo-metrizable locally convex space is a Mackey space.*

Note that this result also justifies the "old way" of finding the Mackey topology of a non locally convex F-space.

Proof of Theorem 1. (cf. [12, Prop. 3, page 641]). By a *local base* for a vector topology we mean a base for the neighborhoods of zero. For a subset A of E let $\tilde{A}$ denote the E-closure of the absolutely convex hull of E. Let μ denote the Mackey topology of E. We will be done if we can show that the family of sets

$$\{\tilde{U} : U \text{ a neighborhood of zero in } E\}$$

is a local base for μ. It is easy to check [7; Theorem 5.1, page 34] that this family is a local base for *some* vector topology τ on E, which is clearly weaker than the original topology. Since E is metrizable its topology has a countable local base $\{U_n\}$, hence $\{\tilde{U}_n\}$ is a countable local base for τ. Thus τ is pseudo-metrizable [7, Theorem 6.7, page 48], hence the above Proposition guarantees that (E, τ) is a Mackey space. To finish the proof, then, we need only show that E' is the τ-dual of E.

Every τ-continuous linear functional is in E' since τ is weaker than the E-topology. Conversely, suppose $\lambda \in E'$. Then λ is bounded on an E-neighborhood of zero, hence on its absolutely convex hull (by linearity of λ), and finally on the E-closure of this absolutely convex hull (by E-continuity of λ). Condition (2) insures that this last set contains a τ-neighborhood of zero, so λ is bounded on a τ-neighborhood of zero, hence τ-continuous. This completes the proof.

The rest of this section contains results about Hardy and Bergman spaces that will be needed later on. For f analytic in U we use the notation

$$M_p^p(f; r) = \int_T |f(r\omega)|^p \, dm(\omega).$$

It is well known that $M_p(f; r)$ increases with r on the interval $[0, 1)$.

The result of Hardy and Littlewood mentioned in Section 1 follows by taking $q = l = 1$ in the following more general theorem.

THEOREM A [5; Theorem 31, pp. 411–412]. *If $0 < p < q$, $l \geq p$, and $\alpha = p^{-1} - q^{-1}$, then there exists a constant $K = K(p, q, l)$ such that*

$$\int_0^1 M_q^l(f; r)(1 - r)^{l\alpha - 1} \, dr \leq K \, \|f\|_p^l$$

for each f in H^p.

COROLLARY A. *For $0 < p < 1$ there exists $K = K(p) > 0$ such that*

$$\|f\|_{1, 1/p-2} \leq K \, \|f\|_p$$

for all $f \in H^p$.

We will also require a simple growth estimate for functions in A_α^p.

THEOREM B. *For $\alpha > -1$ and $0 < p < \infty$ there exists a constant $C = C(\alpha, p)$ such that*

$$M_p(f; r) \leq C \, \|f\|_{p,\alpha} \, (1 - r)^{-(\alpha+1)/p}$$

for each $f \in A_\alpha^p$ and $0 \leq r < 1$.

Proof. Since $M_p(f; r)$ increases on $0 \leq r < 1$ it is enough to prove the result for $\frac{1}{2} \leq r < 1$. For this range of r we have:

$$\begin{aligned} \|f\|_{p,\alpha}{}^p &\geq \int_r^1 M_p{}^p(f; \rho)(1 - \rho)^\alpha \rho \, d\rho \\ &\geq 2^{-1} M_p{}^p(f; r) \int_r^1 (1 - \rho)^\alpha \, d\rho \\ &= (2_\alpha + 2)^{-1} M_p{}^p(f; r)(1 - r)^{\alpha+1}, \end{aligned}$$

where the monotonicity fo $M_p(f; r)$ justifies the second inequality. This completes the proof.

From this estimate and another result of Hardy and Littlewood [5; Theorem 27, page 406] we obtain:

COROLLARY B. *For $0 < p < \infty$ and $\alpha > -1$ there exists $C = C(p, \alpha) > 0$ such that*

$$|f(z)| \leq C \, \|f\|_{p,\alpha} \, (1 - |z|)^{-(\alpha+2)/p}$$

for every f in $A_\alpha{}^p$.

It follows from this Corollary that for each z in U the linear functional

$$f \to f(z) \qquad (f \text{ in } A_\alpha{}^p)$$

is continuous on $A_\alpha{}^p$; and that the topology of $A_\alpha{}^p$ is stronger than the topology κ of uniform convergence on compact subsets of U. The completeness of $A_\alpha{}^p$ follows quickly from this last fact. Corollary B also shows that every bounded subset of $A_\alpha{}^p$ is a normal family, hence the unit ball of $A_\alpha{}^p$ (being κ-closed by Fatou's Lemma) is κ-compact. The same results hold for H^p $(0 < p < \infty)$, and follow from similar inequalities (replace α by -1 in Corollary B [1; page 36]).

To handle the spaces $A_\alpha{}^p$ we need a result on fractional integration. Suppose $\beta > 0$ and $f(z) = \sum a_n z^n$ is analytic on U. Following Duren, Romberg, and Shields [2] we define the *fractional integral* of f of order β to be

$$f_{[\beta]}(z) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n + 1 + \beta)} a_n z^n:$$

it is easily seen to be analytic in U. As noted in [2] this definition differs from the original one given by Hardy and Littlewood in [5] by a factor of z^β; in particular

$$f_{[1]}(z) = z^{-1} \int_0^z f(z) \, ds. \tag{2.1}$$

The inequality we require is due to Duren, Romberg, and Shields.

THEOREM C [2; Theorem 5, page 43]. *For $\alpha > -1$ and $\beta > 0$ there exists $C = C(\alpha, \beta) > 0$ such that*

$$C^{-1} \, \|f_{[\beta]}\|_{1,\alpha} \leq \|f\|_{1,\alpha+\beta} \leq C \, \|f_{[\beta]}\|_{1,\alpha}$$

for every f analytic in U.

Finally we record a pair of standard estimates:

THEOREM D. *Suppose $\alpha > -1$ and $\gamma > 1 + \alpha$. Then there exists $C = C(\alpha, \gamma) > 0$ such that for $0 \leq \rho < 1$:*
(a) [1; *Chapter* 4, *sec.* 6]

$$\int_T |1 - \rho\omega|^{-\alpha} \, dm(\omega) \leq C(1 - \rho)^{1-\alpha}$$

(here C depends only on α, of course),
(b) [14; *Lemma* 6, *page* 291]

$$\int_0^1 (1 - \rho r)^{-\gamma}(1 - r)^\alpha \, dr \leq C(1 - \rho)^{1+\alpha-\gamma}.$$

3. The Mackey topology of H^p $(0 < p < 1)$. In this section we give a proof based on Theorem 1 of the following result of Duren, Romberg, and Shields:

THEOREM 2 [2; Theorems 1 and 7]. *The Mackey topology on H^p is induced by the norm $\|\cdot\|_{1,1/p-2}$ $(0 < p < 1)$.*

Proof. Let $\sigma = 1/p - 2$. By Corollary A the norm $\|\cdot\|_{1,\sigma}$ is defined on H^p, and induces a topology τ weaker than the original one. Thus condition (1) of Theorem A holds, so we need only verify condition (2).

A calculation with power series shows that for any $\beta > -1$ the function

$$K(z, \zeta) = (\beta + 1) \frac{(1 - |\zeta|^2)^\beta}{(1 - \bar{\zeta}z)^{\beta+2}} \tag{3.1}$$

is a reproducing kernel for U; that is,

$$f(z) = \int_U K(z, \zeta)f(\zeta) \, d\lambda(\zeta) \qquad (z \in U) \tag{3.2}$$

for all f analytic in U and integrable with respect to the measure $(1 - |\zeta|^2)^\beta d\lambda(\zeta)$. Fix $\beta > \sigma$. Then in particular (3.2) holds for all f in A_σ^1, hence (by Corollary A) for all f in H^p. For $z, \zeta \in U$ let

$$J(\zeta)(z) = J(z, \zeta) = (1 - |\zeta|)^{-\sigma}K(z, \zeta). \tag{3.3}$$

Each $J(\zeta)$ is an analytic function (of z) in a neighborhood of the closed unit disc, hence a member of H^p. We claim that the collection of all such functions is a bounded subset of H^p; that is,

$$M = \sup\{\|J(\zeta)\|_p : \zeta \in U\} < \infty. \tag{3.4}$$

To prove this, first note that the definition of J quickly yields for $0 \leq r < 1$:

$$\int_T |J(\zeta)(r\omega)|^p \, dm(\omega) \leq C_\beta^p(1 - |\zeta|)^{(\beta+2)p-1}I(r, \zeta)$$

where $C_\beta = 2^\beta(\beta + 1)$, and

$$I(r, \zeta) = \int_T |1 - \bar{\zeta} r\omega|^{-(\beta+2)p}\, dm(\omega).$$

Now $(\beta + 2)p > 1$ (since $\beta > \sigma = 1/p - 2$), so part (a) of Theorem D yields

$$I(r, \zeta) \le C(1 - |\zeta|\, r)^{1-(\beta+2)p} \le C(1 - |\zeta|)^{1-(\beta+2)p}$$

where C is a positive constant independent of $\zeta \in U$ and $0 \le r < 1$. Thus

$$\begin{aligned} ||J(\zeta)||_p^p &= \sup_{0 \le r < 1} \int_T |J(\zeta)(r\omega)|^p\, dm(\omega) \\ &\le C_\beta^p C \end{aligned}$$

which completes the proof of (3.4).

We are going to show that each f in H^p with $||f||_{1,\sigma} \le 1$ lies in the H^p-closure of the absolutely convex hull of the ball

$$V = \{f \in H^p : ||f||_p \le M\},$$

for M given by (3.4). As in our discussion of l^p (section 1), this will show that the H^p-closure of the absolutely convex hull of *every* H^p-ball contains a τ-neighborhood of zero, establishing condition (2) of Theorem 1 and completing the proof.

So let $f \in H^p$ with $||f||_{1,\sigma} \le 1$. Then we can rewrite (3.2) as

$$f(z) = \int J(z, \zeta)\, d\mu(\zeta) \qquad (z \in U) \tag{3.2'}$$

where

$$d\mu(\zeta) = (1 - |\zeta|)^\sigma f(\zeta) d\lambda(\zeta)$$

is a complex Borel measure on U of total variation $||f||_{1,\sigma} \le 1$. Now (3.2′) expresses f as a sort of generalized absolutely convex combination of functions $J(\zeta)$, which lie in V; so to complete the proof we must interpret the right side of (3.2′) as an H^p-limit of honest absolutely convex combinations of elements of V.

For $0 \le r < 1$ let $f_r(z) = f(rz)$. Then $f_r \in H^p$, and it is a standard fact that $f_r \to f$ in H^p as $r \to 1$—[1; Theorem 2.6, page 21]. So we need only show that for each fixed $0 < r < 1$ the function f_r is an H^p-limit of absolutely convex combinations of members of V. Fix $0 < r < 1$. From (3.2′) we have

$$f_r(z) = \int J(rz; \zeta)\, d\mu(\zeta) \qquad (z \in U). \tag{3.5}$$

An easy calculation shows that the function

$$(z, \zeta) \to J(rz, \zeta)$$

is uniformly continuous on $U \times U$ (the fact that $\beta > \sigma$ again enters here), so given $\epsilon > 0$ we can partition U into a finite number of disjoint semi-open polar coordinate rectangles $U_1, \cdots, U_n$ such that for each $z \in U$:

$$\sup\{|J(rz, \zeta) - J(rz, \zeta')| : \zeta, \zeta' \in U_k\} < \epsilon \tag{3.6}$$

for $k = 1, 2, \cdots, n$. Choose $\zeta_k \in U_k$ (independent of z) and set

$$S_\epsilon(z) = \sum_{k=1}^{n} J(rz, \zeta_k)\mu(U_k)$$

for $z \in U$. Since

$$\sum |\mu(U_k)| \leq |\mu| (U) \leq 1,$$

S_ϵ is an absolutely convex combination of functions $J_r(\zeta)$ defined by

$$J_r(\zeta)\ (z) = J(rz, \zeta).$$

Now (3.4) and the definition of the H^p "norm" show that $\|J_r(\zeta)\|_p \leq M$ for each $\zeta \in U$, hence each $J_r(\zeta) \in V$. Thus S_ϵ is an absolutely convex combination of elements of V, so it only remains to show that

$$\lim_{\epsilon \to 0+} \|f_r - S_\epsilon\|_p = 0. \tag{3.7}$$

In fact, it follows from (3.5) that for each $z \in U$:

$$\begin{aligned} |f_r(z) - S_\epsilon(z)| &= \left| \sum_{k=1}^{n} \int_{U_k} [J(rz, \zeta) - J(rz, \zeta_k)]\, d\mu(\zeta) \right| \\ &\leq \sum_{k=1}^{n} \int_{U_k} |J(rz, \zeta) - J(rz, \zeta_k)|\, d\, |\mu|\, (\zeta) \\ &\leq \epsilon \sum_{k=1}^{n} |\mu|(U_k) \quad \text{(from (3.6))} \\ &= \epsilon\, |\mu|\, (U) \\ &\leq \epsilon. \end{aligned}$$

Thus $S_\epsilon \to f_r$ uniformly on U, hence in H^p; and the proof is complete.

Essentially the same proof characterizes the Mackey topology of $H^p(U^n)$ for $0 < p < 1$: here are the essential points. Let U^n, T^n, $d\lambda_n$, and dm_n denote the n-fold products of U, T, $d\lambda$, and dm respectively. For $0 < p < \infty$ the Hardy space $H^p(U^n)$ is the collection of complex valued functions f analytic on U^n such that

$$\|f\|_p^p = \sup_{0 \leq r < 1} \int_{T^n} |f(r\omega)|^p\, dm_n(\omega) < \infty$$

(see [9; Chapter 3] for basic material on these spaces); and the weighted Bergman space $A_\alpha^p(U^n)$ ($\alpha > -1$) is the family of analytic functions f on U^n for which

$$\|f\|_{p,\alpha}^p = \int_{U^n} |f(z)|^p\, w(z)^\alpha\, d\lambda_n(z) < \infty,$$

where

$$w(z) = \prod_{j=1}^{n} (1 - |z_j|) \quad \text{for} \quad z = (z_1, z_2, \cdots, z_n) \in U^n.$$

With this notation Theorem A remains true for f in $H^p(U^n)$. This was proved by Arlene Frazier in [4]: it is *not* a simple deduction from the one variable case. So condition (1) of Theorem 1 is satisfied with $E = H^p(U^n)$ and τ the topology induced on $H^p(U^n)$ by the norm $\|\cdot\|_{1,1/p-2}$ $(0 < p < 1)$. We are going to show that condition (2) is also satisfied, which yields:

THEOREM 2′: *Theorem 2 remains true if H^p and $A_{1/p-2}^1$ are replaced by their counterparts on U^n.*

Proof. Let

$$J_n(\zeta)(z) = J_n(z, \zeta) = \prod_{k=1}^{n} J(z_k, \zeta_k)$$

for $z = (z_1, \cdots, z_n)$ and $\zeta = (\zeta_1, \cdots, \zeta_n) \in U^n$, where J is given by (3.3) and (3.1) with $\beta > \sigma = 1/p - 2$. Fubini's theorem and (3.4) yield

$$\sup\{\|J_n(\zeta)\|_p : \zeta \in U^n\} < \infty,$$

while for $f \in H^p(U^n)$, Fubini's theorem and (3.2′) give the integral representation

$$f(z) = \int_{U^n} J_n(z, \zeta)\, d\mu(\zeta)$$

where

$$d\mu(\zeta) = f(\zeta) w(\zeta)^\sigma d\lambda_n(\zeta).$$

The rest of the proof goes exactly as before: we omit the details.

4. The weighted Bergman spaces. Using the methods of the last two sections we determine the Mackey topology of A_α^p when $0 < p < 1$.

THEOREM 3. *For $\alpha > -1$ and $0 < p < 1$ let $\sigma = (\alpha + 2)/p - 2$. Then $A_\alpha^p \subset A_\sigma^1$, and the Mackey topology of A_α^p is induced by the norm of A_σ^1.*

Note that Theorem 2 can be regarded as the limiting case $\alpha = -1$ of this result. The case $\alpha = 0$ appeared in [10] where it was obtained by calculating dual spaces.

Proof of Theorem 3. Just as in the last section we will verify the hypotheses of Theorem 1, this time with $E = A_\alpha^p$ and τ the topology induced by the A_σ^1 norm.

Condition (1). We need to show that there is a positive constant $C = C(p, \alpha)$ such that

$$\|f\|_{1,\sigma} \le C\, \|f\|_{p,\alpha} \tag{4.1}$$

for each $f \in A_\alpha^p$. As in section 3, this requires Corollary A, but now we need to do some extra work. Fix $f \in A_\alpha^p$ and $0 < \rho < 1$. In what follows, the symbol "C" denotes a positive constant, possibly different at each occurrence, but always independent of f and ρ. From Theorem C and Corollary A respectively, we have:

$$\int_U |f_{[1/p-2]}(z)| \, d\lambda(z) \leq C \, ||f||_{1,1/p-2} \leq C \, ||f||_p \, . \tag{4.2}$$

Let $g = f_{[1/p-2]}$. Then a change of variable, inequality (4.2), and Theorem B respectively yield

$$\begin{aligned} \rho^{-1} \int_0^\rho M_1(g; u) \, du &= \int_0^1 M_1(g; r\rho) \, dr \\ &\leq CM_p(f; \rho) \\ &\leq CM_p^p(f; \rho)(1 - \rho)^{(p-1)(\alpha+1)/p} \, ||f||_{p,\alpha}^{1-p} \, . \end{aligned}$$

Multiplying through by the appropriate power of $1 - \rho$ we obtain

$$(1 - \rho)^{(\alpha+1)/p+1}\left\{\rho^{-1} \int_0^\rho M_1(g; u) \, du\right\} \leq C \, ||f||_{p,\alpha}^{1-p} M_p^p(f; \rho)(1 - \rho)^\alpha. \tag{4.3}$$

We claim that the quantity in braces on the left side of (4.3) dominates $M_1(g_{[1]}; \rho)$. Indeed:

$$\begin{aligned} M_1(g_{[1]}; \rho) &= \int_T |g_{[1]}(\rho\omega)| \, dm(\omega) \\ &= \int_T \left|(\rho\omega)^{-1} \int_0^{\rho\omega} g(s) \, ds\right| \, dt \quad (by\ (2.1)) \\ &\leq \rho^{-1} \int_T \int_0^\rho |g(u\omega)| \, du \, dm(\omega) \\ &= \rho^{-1} \int_0^\rho M_1(g; u) \, du \quad \text{(by Fubini)}. \end{aligned}$$

Thus (4.3) yields

$$M_1(g_{[1]}; \rho)(1 - \rho)^{(\alpha+1)/p-1} \leq C \, ||f||_{p,\alpha}^{1-p} M_p^p(f; \rho)(1 - \rho)^\alpha \, .$$

Integrating both sides of this inequality with respect to ρ over [0, 1), and applying Theorem C to the resulting right hand side, we obtain

$$||g||_{1,(\alpha+1)/p} \leq C \, ||g_{[1]}||_{1,(\alpha+1)/p-1} \leq C \, ||f||_{p,\alpha} \, .$$

But $g = f_{[1/p-2]}$, so using Theorem C on this last inequality:

$$||f||_{1,(\alpha+2)/p-2} \leq C \, ||f_{[1/p-2]}||_{1,(\alpha+1)/p} \leq C \, ||f||_{p,\alpha}$$

which is the desired result.

Condition (2). Suppose $f \in A_\alpha^p$ and $||f||_{1,\sigma} \leq 1$. As in section 3 we need only show that f lies in the A_α^p-closure of the absolutely convex hull of a bounded subset of A_α^p.

Choose $\beta > \sigma$ and define $J(\zeta)(z) = J(z, \zeta)$ by (3.3) and (3.1), so we have the representation (3.2′) for all f in A_α^p and z in U. As in section 3:

$$M = \sup \{||J(\zeta)||_{p,\alpha} : \zeta \in U\} < \infty. \tag{4.4}$$

To prove this, fix $\zeta \in U$ and calculate:

$$\begin{aligned} ||J(\zeta)||_{p,\alpha}^p &= \int_U |J(z, \zeta)|^p (1 - |z|)^\alpha \, d\lambda(z) \\ &= (1 - |\zeta|)^{2p-(\alpha+2)} \int_U |K(z, \zeta)|^p (1 - |z|)^\alpha \, d\lambda(z) \\ &\leq C_\beta^p (1 - |\zeta|)^\gamma I(\zeta), \end{aligned}$$

where $\gamma = (2 + \beta)p - (\alpha + 2)$, $C_\beta = 2^\beta(\beta + 1)$, and

$$\begin{aligned} I(\zeta) &= \int_U \frac{(1 - |z|)^\alpha}{|1 - \bar{\zeta} z|^{(\beta+2)p}} \, d\lambda(z) \\ &= \int_0^1 (1 - r)^\alpha \left\{ \int_T \frac{dm(\omega)}{|1 - \bar{\zeta} r\omega|^{(\beta+2)p}} \right\} r\,dr \\ &\leq C \int_0^1 \frac{(1 - r)^\alpha \, dr}{(1 - |\zeta| r)^{(\beta+2)p-1}}, \end{aligned}$$

where the last inequality follows from part (a) of Theorem D (section 2), because $(\beta + 2)p - 1 > 1 + \alpha > 0$. Applying part (b) of Theorem D to the last integral, we obtain

$$I(\zeta) \leq C(1 - |\zeta|)^{-\gamma},$$

hence

$$\begin{aligned} ||J(\zeta)||_{p,\alpha}^p &\leq C_\beta^p (1 - |\zeta|^\gamma) \, I(\zeta) \\ &\leq C_\beta^p C \end{aligned}$$

where C is independent of $\zeta \in U$. This proves (4.4).

The rest of the proof proceeds just as in section 3; and we omit the details. Again we need to know that $f_r \to f$, this time in A_α^p; but this follows quickly from the fact that $M_p(f\,; r)$ increases on $0 \leq r < 1$, and the Lebesgue Dominated Convergence Theorem.

5. The diagonal map. For f a complex valued function on U^n we define

$$\Delta f(z) = f(z, z, \cdots, z)$$

for z in U, and call Δ the *diagonal map*. Clearly Δ takes functions analytic on U^n to functions analytic on U. Horowitz and Oberlin [6] ,extending previous work of Rudin [9; pages 53 and 62] have shown that Δ is a continuous linear transformation taking $H^p(U^n)$ *onto* A_{n-2}^p for $1 \leq p < \infty$. Duren and Shields [3] have shown that even when $0 < p < 1$ the diagonal map takes $H^p(U^n)$ continuously into A_{n-2}^p, but it is not known if Δ is *onto* in this case.

We are going to prove some results which connect this problem with the work of the previous sections. Although they do not prove that $\Delta\{H^p(U^n)\} = A_{n-2}^p$ for $0 < p < 1$, these results tend to support the conjecture. In addition they yield short proofs of the "onto-ness" of $\Delta : H^1(U^n) \rightarrow A_{n-2}^1$.

Our first result appears implicitly in [6; page 770, part (c)]. To state it we need the notion of the *absolutely p-convex hull* of a set S. This is the collection of finite sums $\sum \lambda_i s_i$ where $s_i \in S$ and the λ_i are scalars with $\sum |\lambda_i|^p \leq 1$ $(0 < p < 1)$. A set is *absolutely p-convex* if it coincides with its absolutely p-convex hull. For $p = 1$ we get the usual notion of absolute convexity. Clearly linear transformations preserve absolute p-convexity.

THEOREM 4. *For $0 < p \leq 1$ the image under Δ of the unit ball of $H^p(U^n)$ contains the closure in A_{n-2}^p of the absolutely p-convex hull of the $H^{p/n}$ unit ball.*

To see how this relates to the previous work, consider the case $p = 1$. We know from (the proof of) Theorem 2 that for some $\delta > 0$ the set

$$S = H^{1/n} \text{ closure of the absolutely convex hull of the } H^{1/n} \text{ unit ball}$$

contains every function in $H^{1/n}$ with $||f||_{1,n-2} \leq \delta$. Thus the closure in A_{n-2}^1 of S contains the ball of radius δ, so by Theorem 4 the Δ-image of the $H^{1/n}$ unit ball contains an A_{n-2}^1-ball. Thus Δ takes $H^1(U^n)$ onto A_{n-2}^1, providing an easy proof of the result of Horowitz and Oberlin for the case $p = 1$ (in this case their original proof required a complicated duality argument).

The same reasoning would yield the onto-ness of $\Delta : H^p(U^n) \rightarrow A_{n-2}^p$ for $0 < p < 1$ if we could prove the following

CONJECTURE. *For $0 < p < 1$ the closure in A_{n-2}^p of the p-convex hull of the $H^{p/n}$ unit ball contains an A_{n-2}^p ball ($n = 2, 3, \cdots$).*

Unfortunately the methods of section 3 do not seem to apply when $p < 1$. The difficulty lies in trying to interpret the integral in (3.2) as an absolutely p-convex combination; and we have no substitute to offer. The "converse" result—that the A_{n-2}^p closure of the absolutely p-convex hull of the $H^{p/n}$ unit ball lies in some A_{n-2}^p ball—is true, and follows from Theorem A by replacing p with p/n and setting $q = l = p$ $(0 < p < 1, n = 2, 3, \cdots)$.

Proof of Theorem 4. For $f \in H^{p/n}$ the classical factorization theory [1; section 2.4] allows us to write

$$f = f_1 f_2 , \cdots , f_n$$

where $f_j \in H^p$ and $\|f_j\|_p = \|f\|_{p/n}^{1/n}$ $(1 \leq j \leq n)$. So letting

$$F(z_1, z_2, \cdots, z_n) = f_1(z_1)f_2(z_2) \cdots f_n(z_n)$$

for $z_1, z_2, \cdots, z_n \in U$; we have $F \in H^p(U^n)$ and

$$\|F\|_p = \|f\|_{p/n}, \tag{1}$$

$$\Delta F = f. \tag{2}$$

So the image under Δ of the $H^p(U^n)$ unit ball B contains the $H^{p/n}$ unit ball; and since B is absolutely p-convex, $\Delta(B)$ actually contains the absolutely p-convex hull of the $H^{p/n}$ unit ball. Now $\Delta : H^p(U^n) \to A_{n-2}^p$ remains continuous when both spaces have their respective topologies of uniform convergence on compact sets, and since B is compact in this topology (see section 2), so is $\Delta(B)$. Since this topology is weaker on A_{n-2}^p than the original one, we see that $\Delta(B)$ is closed in A_{n-2}^p, which completes the proof of Theorem 4.

In [6] Horowitz and Oberlin prove that Δ takes $H^p(U^n)$ *onto* A_{n-2}^p for $1 < p < \infty$ by using the Cauchy kernel to construct an operator

$$S : A_{n-2}^p \to H^p(U^n)$$

such that ΔS is the identity operator on A_{n-2}^p. However this did not work for $p = 1$, where they had to employ a duality argument. The next result shows that their method for $1 < p < \infty$ can be made to work for $H^1(U^n)$, and even for $A_\alpha^1(U^n)$, if the Cauchy kernel is replaced by one of the sort used in proving Theorems 2 and 3.

THEOREM 5. *For $0 < p < 1$ and $n = 2, 3, 4, \cdots$; choose $\beta > n/p - 2$ and let*

$$J(z_1, z_2, \cdots, z_n; \zeta) = (\beta + 1)(1 - |\zeta|^2)^\beta \prod_{j=1}^{n} (1 - \bar{\zeta} z_j)^{-(\beta+2)/n}$$

for $z_1, z_2, \cdots, z_n, \zeta \in U$. For $f \in A_{n/p-2}^1$ set

$$Sf(z_1, z_2, \cdots, z_n) = \int_U f(\zeta) J(z_1, z_2, \cdots, z_n; \zeta)\, d\lambda(\zeta).$$

Then $Sf \in A_{1/p-2}^1(U^n)$, and ΔS is the identity map on $A_{n/p-2}^1$. If $p = 1$ the same is true with $H^1(U^n)$ replacing $A_{1/p-2}^1(U^n)$.

Proof. We consider only the case $0 < p < 1$. The proof for $p = 1$ is essentially the same (in fact, easier). So fix $f \in A_{n/p-2}^1$. Because of the choice of β, the integral in the definition of Sf converges absolutely for each $(z_1, \cdots, z_n) \in U^n$, and Sf is analytic on U^n. By (3.1) and (3.2) we have for each $z \in U$:

$$\Delta(Sf)(z) = \int_U f(\zeta) K(z, \zeta)\, d\lambda(\zeta) = f(z)$$

so ΔS is the identity on $A_{n/p-2}^1$; and it only remains to show that $Sf \in A_{1/p-2}^1(U^n)$.

For convenience let $\sigma = 1/p - 2$. Then, using the multivariable notation introduced at the end of section 3:

$$(5.1) \qquad \|Sf\|_{1,\sigma} = \int_{U^n} |Sf(z)|\, w(z)^\sigma \, d\lambda_n(z)$$
$$\leq \int_U \left\{ \int_{U^n} |J(z, \zeta)|\, w(z)^\sigma \, d\lambda_n(z) \right\} |f(\zeta)|\, d\lambda(\zeta)$$
$$= (\beta + 1) \int_U I(\zeta)^n (1 - |\zeta|^2)^\beta \, |f(\zeta)|\, d\lambda(\zeta)$$

where

$$I(\zeta) = \int_U (1 - |z|)^\sigma \, |1 - \bar{\zeta} z|^{-(\beta+2)/n} \, d\lambda(z)$$
$$= \int_0^1 (1 - r)^\sigma \left\{ \int_T |1 - \bar{\zeta} r\omega|^{-(\beta+2)/n} \, dm(\omega) \right\} r \, dr.$$

Since $\beta > n/p - 2$, we have $(\beta + 2)/n > 1/p$, from which it follows that both $(\beta + 2)/n$ and $(\beta + 2)/n - 1 - \sigma$ are > 1. Thus parts (a) and (b) of Theorem D can be applied successively to $I(\zeta)$, yielding

$$I(\zeta) \leq C \int_0^1 (1 - r)^\sigma (1 - |\zeta| r)^{1-(\beta+2)/n} \, dr$$
$$\leq C(1 - |\zeta|)^{2+\sigma-(\beta+2)/n}$$
$$= C(1 - |\zeta|)^{1/p-(\beta+2)/n}$$

where C is independent of $\zeta \in U$. Upon substituting this estimate into (5.1) we obtain

$$(5.2) \qquad \|Sf\|_{1,\sigma} \leq 2^\beta C^n (\beta + 1) \int_U |f(\zeta)|\, (1 - |\zeta|)^{n/p-2} \, d\lambda(\zeta)$$

which completes the proof.

COROLLARY. *For $0 < p < 1$ and $n = 2, 3, 4, \cdots$, the diagonal map is a bounded linear operator taking $A_{1/p-2}^1(U^n)$ onto $A_{n/p-2}^1$.*

The "onto-ness" of Δ follows immediately from Theorem 5. The fact that Δ takes $A_{1/p-2}^1(U^n)$ continuously *into* $A_{n/p-2}^1$ is a consequence of the following result.

PROPOSITION. *Let X and Y be F-spaces with separating duals, and let $\hat{X}$ and $\hat{Y}$ be their respective Mackey completions. Suppose T is a continuous linear map taking X into Y. Then there is a unique extension $\hat{T}$ of T to a continuous linear map taking $\hat{X}$ into $\hat{Y}$. If in addition $T(X) = Y$, then $\hat{T}(\hat{X}) = \hat{Y}$.*

Before proving this, let us finish the proof of the Corollary. Recall that for $0 < p < 1$ the space $A_{1/p-2}^1(U^n)$ contains $H^p(U^n)$ and induces the Mackey topology on it (end of section 3), while for $n > 1$ the space $A_{n/p-2}^1$ contains A_{n-2}^p and induces its Mackey topology (Theorem 3, section 4). Moreover the proofs we gave of these results show that each of the smaller spaces is dense in its containing space (it is easy to see this directly: the polynomials are dense in each space). Thus $A_{1/p-2}^1(U^n)$ can be identified with the Mackey completion of $H^p(U^n)$, in the sense that there is a linear homeomorphism from $A_{1/p-2}^1(U^n)$ onto $[H^p(U^n)]^\wedge$ that restricts to the identity of $H^p(U^n)$; and the same is true of $A_{n/p-2}^1$ and A_{n-2}^p. Thus the Proposition insures that the diagonal map Δ, which is known to take $H^p(U^n)$ continuously into A_{n-2}^p, extends to a bounded linear operator $\hat{\Delta}$ taking $A_{1/p-2}^1(U^n)$ into $A_{n/p-2}^1$. It is easy to check that $\hat{\Delta}$ is still the diagonal map, which completes the proof of the Corollary.

This argument also shows that the Corollary can be restated as follows:

> *For $n > 1$ and $0 < p < 1$ the diagonal map takes the Mackey completion of $H^p(U^n)$ onto that of A_{n-2}^p.*

The last part of the Proposition shows that the Corollary, viewed in this way, is actually a weaker form of the conjecture that Δ takes $H^p(U^n)$ onto A_{n-2}^p $(0 < p < 1, n > 1)$.

Proof of the Proposition. We denote the convex hull of a set S by conv S. The argument used to prove Theorem 1 shows that if E is an F-space, then the family of sets

$$\{\text{conv } U\colon U \text{ an } E\text{-neighborhood of zero}\}$$

is a local base for the Mackey topology of E (see also [12; Prop. 3, page 641]).

Now suppose $T : X \to Y$ is continuous. Then for each neighborhood V of 0 in Y there is a neighborhood U of 0 in X such that $T(U) \subset V$. Since T is linear it follows that T (conv V) $\subset$ conv V, so (by the last paragraph) T is continuous even when X and Y have their Mackey topologies. Thus T has a unique extension to a continuous linear transformation $\hat{T} : \hat{X} \to \hat{Y}$.

If, in addition, $T(X) = Y$, then T is an open mapping [7; Theorem 11.4, page 99]. By the description of the Mackey topology given in the first paragraph, T remains open when X and Y have their Mackey topologies.

Now let $\{U_n\}_1^\infty$ be a sequence of Mackey neighborhoods in X such that whenever $x_n \in U_n$ $(1 \le n < \infty)$, the partial sums of the series $\sum_1^\infty x_n$ form a Cauchy sequence in the Mackey topology of X (for example, fix a metric d for the Mackey topology of X, and let U_n be the d-ball of radius 2^{-n}). By the open-ness of T there exists a Mackey neighborhood V_n of zero in Y such that $T(U_n) \supset V_n$ $(1 \le n < \infty)$.

Now suppose $\hat{y} \in \hat{Y}$. Then there is a sequence $(y_n)_0^\infty$ in Y such that

$$\hat{y} = \sum_{n=0}^{\infty} y_n \tag{1}$$

where the series converges in $\hat{Y}$, and

$$(2) \qquad y_n \in V_n \quad \text{for} \quad n \geq 1.$$

Choose $x_n \in X$ such that $Tx_n = y_n$ for all $n \geq 0$, making sure that $x_n \in U_n$ for $n \geq 1$. Then the series $\sum_{n=0}^{\infty} x_n$ converges to an element $\hat{x} \in \hat{X}$, and by the continuity of $\hat{T}$:

$$\hat{T}\hat{x} = \sum_{n=0}^{\infty} \hat{T}x_n = \sum_{n=0}^{\infty} Tx_n = \sum_{n=0}^{\infty} y_n = \hat{y}.$$

Thus $\hat{T}(\hat{X}) = \hat{Y}$, and the proof is complete.

We note in closing that because ΔS is the identity map on $A_{n/p-2}^1$ $(0 < p < 1)$, we have $(S\Delta)^2 = S\Delta$; hence $S\Delta$ is a bounded projection taking $A_{1/p-2}^1(U^n)$ into itself. Now S is one-to-one (since $S\Delta$ = identity), so the null space of $S\Delta$ coincides with that of Δ: those functions in $A_{1/p-2}^1(U^n)$ which vanish on the diagonal of U^n. Thus the null space of Δ is a closed, *complemented* subspace of $A_{1/p-2}^1(U^n)$. The same is true for $p = 1$, with $H^1(U^n)$ replacing $A_{1/p-2}^1(U^n)$.

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