Spectral Synthesis and Common Cyclic Vectors

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Dedicated to the memory of Allen L. Shields

Introduction

The ancestry of our work traces back to Shields and Wallen's study [26] of the commutant of the operator M_z of multiplication by z on certain Hilbert spaces of functions holomorphic in the unit disc. In addition to a few other natural hypotheses, Shields and Wallen assumed all point evaluation functionals on their spaces to be bounded, and asked at the end of their paper if, under these conditions, the adjoint of M_z must always have a cyclic vector (they also asked this question for M_z itself).

Later, Wogen [30] proved a striking "hypercyclicity theorem" for the Hardy space H^2, which asserts that the collection of adjoints of nonscalar multiplication operators on H^2 has a common cyclic vector ("nonscalar" means "not a constant multiple of the identity operator"). Finally, Chan [6] recently extended Wogen's theorem to Hilbert spaces of functions, holomorphic on plane domains, that obey hypotheses similar to those imposed by Shields and Wallen (in fact, Chan's work takes place in a Banach space setting).

Chan's methods, which refine those of Wogen, require that (besides the natural assumption of continuity of point evaluations) some additional hypotheses must be placed on the space: In particular, the space must contain the constant functions and be invariant under multiplication by z. The latter hypothesis rules out, for example, the Dirichlet space of certain complicated but still simply connected plane domains [1, Thms. 1 and 10]. In addition, Chan's methods require a "division hypothesis": If α is a point of the domain on which the space lives, and a function in the space vanishes at α, then its quotient by $z-\alpha$ must also belong to the space.

The aim of this paper is to remove all these extra hypotheses, thereby placing Wogen's theorem in its natural setting, and answering the most general version of Shields and Wallen's adjoint cyclicity question.

MAIN THEOREM. *Suppose H is a Hilbert space of functions holomorphic on a plane domain Ω, and suppose that for each point $\lambda \in \Omega$ the linear functional of evaluation at λ is bounded on H. Then there is a vector in*

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H that is cyclic for the adjoint of every nonscalar multiplication operator on H.

The hypothesis about point evaluations amounts to requiring that the identity map take H continuously into the space of all functions, holomorphic on Ω, endowed with the topology of uniform convergence on compact subsets of Ω (see the remark following Lemma 1.1). The reader should note that nothing else is demanded of H; it need not, for example, separate points of Ω, or contain the constant functions, or admit multiplication by the independent variable z.

Furthermore, a little twist in the proof, indicated at the end of Section 4, serves to establish the Main Theorem for arbitrary domains Ω in $\mathbf{C}^N$. This extension to several variables underscores the importance of having a method of proof that does not require division hypotheses of the sort mentioned above.

Finally, our methods also apply to the Frechet space $H(\mathbf{C}^N)$ of entire functions on $\mathbf{C}^N$, where they produce common cyclic vectors for the collection of nonscalar continuous linear operators that commute with translations. Previously Chan had obtained this result for the subclass of partial differential operators with constant coefficients [7].

Our method of proof, which is necessarily quite different from that of Chan and Wogen, evolves from a separate line of inquiry initiated by Wermer [29]. We create the desired cyclic vectors explicitly as infinite linear combinations of reproducing kernels, with the correct choice of kernel functions arising from analysis of a concept similar to Wermer's notion of *spectral synthesis sequence* [29, Thm. 1(iii)]. By contrast, the method of Chan and Wogen involves writing down the cyclic vectors as linear combinations of kernel functions for successive derivatives at a single point.

The third section of this paper contains a detailed discussion of our strategy of proof. Here the necessity of dealing with some form of spectral synthesis becomes clear. In the first two sections we set out the necessary preliminary material, and discuss some examples. In the fourth section we prove the results about spectral synthesis which complete the proof of the Main Theorem. In the final section we use an idea of Godefroy and Shapiro [11] to strengthen the conclusion of our main theorem: There is actually a dense linear manifold, invariant under the adjoint of every multiplication operator on H, which consists (except for zero) entirely of vectors cyclic for every nonscalar adjoint multiplication operator. We outline the proof of our result about operators on spaces of entire functions, and discuss some related work of Clancey and Rogers [8].

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1. Preliminaries

Throughout this paper H will denote a Hilbert space ($\neq\{0\}$) of functions holomorphic on a plane domain Ω such that, for each point $\lambda\in\Omega$, the linear functional of *evaluation at* λ:

$$f\to f(\lambda)\quad (f\in H)$$

is bounded on H.

In this section we collect the elementary results about such spaces that are needed for the sequel. All of this material is well known; it is presented solely for the convenience of the reader.

REPRODUCING KERNELS. By the continuity of point evaluations and by the Riesz representation theorem, there exists for each $\lambda\in\Omega$ a unique function $k_\lambda\in H$ such that

$$f(\lambda)=\langle f,k_\lambda\rangle\quad (f\in H).$$

The function k_λ is the *reproducing kernel* for the point λ, and its norm is the same as that of the corresponding evaluation functional. This fact, along with the boundedness of holomorphic functions on compact sets and the uniform boundedness principle, yields the following.

1.1. LEMMA. *For any compact subset F of* Ω, $\sup\{\|k_\lambda\|:\lambda\in F\}<\infty$.

An immediate consequence of this result is the fact that convergence in H implies uniform convergence on compact subsets of Ω.

MULTIPLIERS. A complex valued function φ on Ω for which $\varphi f\in H$ for every $f\in H$ is called a *multiplier* of H, and the collection of all of these multipliers is denoted $\mathfrak{M}(H)$. Each multiplier φ of H determines a linear multiplication operator M_φ on H by the formula:

$$M_\varphi f=\varphi f\quad (f\in H).$$

The boundedness of point evaluations and the closed graph theorem ensure that M_φ is a bounded operator on H; this in turn leads to a couple of natural restrictions that must be placed on any function that hopes to be a multiplier.

1.2. PROPOSITION. *Each multiplier is a bounded holomorphic function on* Ω.

Proof (cf. [26, Lemma 3, p. 782]). We are assuming that H contains a holomorphic function f that does not vanish identically on Ω. Suppose that $\varphi\in\mathfrak{M}(H)$ and that $\lambda\in\Omega$ is not a zero of f. Then, for each positive integer n,

$$|\varphi(\lambda)|^n|f(\lambda)|=|(M_\varphi^n f)(\lambda)|=|\langle M_\varphi^n f,k_\lambda\rangle|\leq\|M_\varphi^n\|\|f\|\|k_\lambda\|\leq\|M_\varphi\|^n\|f\|\|k_\lambda\|.$$

Now take nth roots on both extremes of the last inequality, let $n\to\infty$, and use the fact that $f(\lambda)\neq 0$. The result is

$$|\varphi(\lambda)| \le \|M_\varphi\|.$$

Thus $|\varphi| \le \|M_\varphi\|$ on $\Omega \backslash f^{-1}\{0\}$, a dense open subset of Ω. Since φ is a multiplier, we have $\varphi f = g \in H$, so $\varphi = g/f$ is holomorphic on $\Omega \backslash f^{-1}\{0\}$ and bounded on that set by $\|M_\varphi\|$. Since f is not identically zero, the set $f^{-1}\{0\}$ has no limit point in Ω, so Riemann's theorem on removable singularities asserts that φ has a holomorphic extension to Ω, which is necessarily also bounded by $\|M_\varphi\|$. □

REMARKS. (i) A closer look at the proof above shows that $|\varphi|$ is actually bounded on Ω by the spectral radius of M_φ.

(ii) Frequently, but not always, $\mathfrak{M}(H)$ consists of *all* bounded holomorphic functions on Ω. We will discuss some examples in the next section.

The following result, which links multipliers with kernel functions, is the starting point for all of our work.

1.3. PROPOSITION. *If* $\varphi \in \mathfrak{M}(H)$ *and* $\lambda \in \Omega$, *then* $M_\varphi^* k_\lambda = \overline{\varphi(\lambda)} k_\lambda$.

Proof (cf. [26, Lemma 4, p. 783]). For each $f \in H$ we have

$$\begin{aligned}\langle M_\varphi^* k_\lambda, f\rangle &= \langle k_\lambda, M_\varphi f\rangle = \langle k_\lambda, \varphi f\rangle \\ &= \overline{\varphi(\lambda) f(\lambda)} = \overline{\varphi(\lambda)}\langle k_\lambda, f\rangle = \langle \overline{\varphi(\lambda)} k_\lambda, f\rangle,\end{aligned}$$

which yields the desired result. □

TERMINOLOGY. A bounded linear operator T on a Hilbert space is *cyclic* if the space has a vector x for which the linear span of the orbit $\{T^n x\}_{n=0}^\infty$ is dense. In this case x is called a *cyclic vector* for T.

A scalar multiple of the identity operator is called a *scalar operator.* All other operators are *nonscalar.* The nonscalar multiplication operators are precisely the ones whose symbols are nonconstant.

2. Some Examples

For simplicity we assume in this section that the domain Ω is bounded, and contains the origin. We discuss some of the most important examples of Hilbert spaces of functions holomorphic on Ω that satisfy the hypotheses of our Main Theorem, and briefly discuss the multipliers of these spaces. The next-to-last example shows how the Main Theorem can apply even when the underlying space of holomorphic functions is not immediately in evidence.

Recall the notation $H(\Omega)$ for the collection of functions holomorphic on Ω, and denote by $H^\infty(\Omega)$ the space of *bounded* holomorphic functions on Ω. Let U denote the open unit disc of the complex plane.

2.1. THE HARDY SPACE H^2 (cf. [9]). This is the Hilbert space of functions f holomorphic on U for which

$$\|f\|^2 = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty.$$

A standard growth estimate shows that for each point of U, the linear functional of evaluation at this point is continuous on H^2 [19, Chap. 17, Prob. 10, p. 382]. It follows immediately from the above definition that every bounded holomorphic function on U is a multiplier of H^2.

For H^2 our Main Theorem reduces to Wogen's original hypercyclicity theorem. It is possible to define a Hardy space $H^2(\Omega)$ for domains Ω other than the unit disc (see, e.g., [9, Chap. 10] and [10]). What results is a Hilbert space in which point evaluations are still continuous, and for which the space of multipliers still coincides with the collection of bounded holomorphic functions on Ω.

2.2. THE BERGMAN SPACE OF Ω. The Bergman space $L_a^2(\Omega)$ is the Hilbert space of functions f holomorphic on Ω for which

$$\|f\|^2 = \int_\Omega |f|^2 \, dA < \infty,$$

where dA represents Lebesgue area measure on the plane. It follows easily from the subharmonicity of $|f|^2$ that point evaluations are bounded on $L_a^2(\Omega)$, so that again our Main Theorem applies. Just as for the Hardy spaces, it is easy to see that $\mathfrak{M}(L_a^2(\Omega)) = H^\infty(\Omega)$.

2.3. THE DIRICHLET SPACE. The Dirichlet space of Ω is the space $\mathfrak{D}(\Omega)$ consisting of functions f holomorphic on Ω for which

$$\|f\|^2 = |f(0)|^2 + \int_\Omega |f'|^2 \, dA < \infty.$$

According to the change of variable formula, the integral on the right is the multiplicity area of $f(\Omega)$. It is not difficult to check that $\mathfrak{D}(\Omega)$ satisfies the hypotheses of our Main Theorem. What is interesting here is that the collection of multipliers of $\mathfrak{D}(\Omega)$, which is by Proposition 1.2 a subset of $H^\infty(\Omega)$, need not coincide with $H^\infty(\Omega)$. This is easiest to see when $\Omega = U$. Each multiplier of $\mathfrak{D}(U)$ actually belongs to $\mathfrak{D}(U)$ (apply the multiplication operator to the constant function 1). But there exist bounded holomorphic functions φ on U that do not belong to $\mathfrak{D}(U)$—for example, the covering map $\exp\{(z+1)/(z-1)\}$ from U onto $U \setminus \{0\}$ is bounded, but its image has infinite multiplicity area, so it does not belong to the Dirichlet space.

The multipliers of $\mathfrak{D}(U)$ have been identified by Stegenga [27], who gives an elegant characterization in terms of a "capacitary Carleson measure" condition.

2.4. WEIGHTED DIRICHLET SPACES. The previous two examples can, of course, be generalized considerably by replacing Lebesgue measure by other measures. When $\Omega = U$, all three previous examples can be subsumed into a single scale of spaces defined in this way. For each $\alpha > -1$, let D_α denote the collection of functions $f \in H(U)$ for which

$$\|f\|^2 = |f(0)|^2 + \int_U |f'(z)|^2 (1-|z|)^\alpha \, dA(z) < \infty.$$

Routine calculations show (cf. [14]) that the spaces D_α increase with α, and that D_α is $\mathfrak{D}(U)$ when $\alpha = 0$, $H^2(U)$ when $\alpha = 1$, and $L_a^2(U)$ when $\alpha = 2$. More generally, for $\alpha > 1$, the space D_α coincides with the Bergman space of U defined with respect to the weighted Lebesgue measure $(1-|z|)^{\alpha-2} \, dA(z)$. Once again, each space D_α obeys the hypotheses of the Main Theorem.

Shields and Wallen give axioms on a Hilbert space of holomorphic functions on U which ensure that the commutant of M_z is precisely the space of multiplication operators [26, Lemma 6, p. 785]. All that is required, in addition to the obvious hypothesis that M_z act boundedly on the space, is that point evaluations be continuous and that the kernel functions be simple eigenfunctions of the adjoint of M_z. It is easy to check that the weighted Dirichlet spaces D_α satisfy these axioms, and hence for these spaces our Main Theorem can be rephrased as follows: *for each $\alpha > -1$ there is a function in D_α that is cyclic for every nonscalar operator that commutes with M_z^*.*

2.5. THE CESÀRO OPERATOR. Let C_0 denote the Cesàro operator defined on the sequence space ℓ^2 by

$$(C_0 f)(n) = \frac{1}{n+1} \sum_{j=0}^{n} f(j) \quad (f \in \ell^2;\ n = 0, 1, 2, \ldots).$$

Shields and Wallen [26] have shown that the operator $1 - C_0$ is unitarily equivalent to the operator M_z of multiplication by z on a Hilbert space H_0 of analytic functions on U that has bounded point evaluations. They also show that the commutant of M_z is the collection of multiplication operators on H_0. Thus our Main Theorem provides a vector in H_0 that is cyclic for every nonscalar operator that commutes with M_z^*. Since an operator commutes with $1 - C_0$ if and only if it commutes with C_0, we may conclude that *there is a vector in ℓ^2 that is cyclic for every nonscalar operator that commutes with C_0^*.*

2.6. DE BRANGES'S SPACES. The previous examples show the advantage of having the conclusion of the Main Theorem available in maximum generality. This need for generality is even more striking in case the Hilbert space H is one of the de Branges spaces $\mathcal{H}(b)$ defined for b holomorphic and of modulus < 1 on the unit disc. We will not formally define these spaces here, except to say that they generalize, in a natural way, those subspaces of H^2 invariant under the adjoint of M_z; through the work of Sarason and others, these spaces are playing an increasingly important role in function theoretic operator theory (see, e.g., [12], [13], [23], [24], [25]).

The point we wish to emphasize is that each space $\mathcal{H}(b)$ is a Hilbert space of holomorphic functions on U that obeys the hypotheses of our Main Theorem (see [23] for details). If b is a nonconstant inner function, then $\mathcal{H}(b)$

is an ordinary M_z^*-invariant subspace of H^2 and has only the constant functions as multipliers ([12], [13]). In the other direction, if b is not an extreme point of the H^∞ unit ball, then every function holomorphic in a neighborhood of the closed unit disc is a multiplier [24, p. 79]. It is not known exactly which b's yield spaces $\mathcal{H}(b)$ with nontrivial multipliers; for example, it is not known if there are extreme points of the H^∞ unit ball for which this happens.

These difficulties notwithstanding, our Main Theorem asserts that if $\mathcal{H}(b)$ does have nontrivial multipliers then the collection of adjoints of these operators has a common cyclic vector.

3. Main Theorem: Strategy of the Proof

Recall the hypothesis of our Main Theorem: *H is a Hilbert space of functions holomorphic on Ω and, for each point $\lambda \in \Omega$, the functional of evaluation at λ is bounded on H.* Our goal is to show that H contains a function that is cyclic for every operator M_φ^*, where φ is a nonconstant multiplier of H.

We begin with an infinite sequence $\{\lambda_j\}$ of distinct points of Ω that converge to some point of Ω. By Lemma 1.1, for each sequence $\{b_j\}$ of complex numbers with $\sum |b_j| < \infty$, the series

$$f = \sum_{j=1}^{\infty} b_j k_{\lambda_j} \tag{1}$$

converges in H to a function $f \in H$. We will find a way of choosing the points $\{\lambda_j\}$ so that f has the desired "universal cyclicity" for every absolutely summable sequence $\{b_j\}$ with infinitely many nonzero terms.

To see what is involved, fix a nonconstant multiplier φ of H and a nonnegative integer n. Apply $(M_\varphi^*)^n$ to both sides of (1), and use the H-convergence of the series, the fact that M_φ^n is the multiplication operator with symbol φ^n, and Proposition 1.3. The result is

$$(M_\varphi^*)^n f = \sum_{j=1}^{\infty} b_j \overline{\varphi(\lambda_j)}^n k_{\lambda_j}.$$

Hence, if $g \in H$ is orthogonal to the M_φ^*-orbit of f then, for each nonnegative integer n,

$$0 = \langle g, (M_\varphi^*)^n f \rangle = \left\langle g, \sum_{j=1}^{\infty} b_j \overline{\varphi(\lambda_j)}^n k_{\lambda_j} \right\rangle = \sum_{j=1}^{\infty} \bar{b}_j g(\lambda_j) \varphi(\lambda_j)^n. \tag{2}$$

Now suppose that the points $\{\lambda_j\}$ may be chosen so that, for every nonconstant multiplier φ of H,

$$\left.\begin{array}{l} \sum |\mu_j| < \infty \text{ and} \\ \sum \mu_j \varphi(\lambda_j)^n = 0 \text{ for all } n \geq 0 \end{array}\right\} \Rightarrow \mu_j = 0 \quad \text{for all sufficiently large } j. \tag{3}$$

Then, if we set $\mu_j = \bar{b}_j g(\lambda_j)$ and assume that infinitely many of the terms b_j are nonzero, we see that g must vanish on a subsequence of $\{\lambda_j\}$. Since any subsequence of $\{\lambda_j\}$ clusters at a point of Ω, this forces g to vanish identically. Thus only the zero function can be orthogonal to the M_φ^*-orbit of f, so that f is cyclic for M_φ^*.

Hence, if we show how to choose points $\{\lambda_j\}$ so that (3) holds for every nonconstant multiplier of H, then the proof of our main theorem will be complete. To see that some work is needed, note that an obvious necessary condition for success is that for each nonconstant multiplier φ, the terms of the sequence $\{\varphi(\lambda_j)\}$ must eventually be distinct. So, for example, if H is the Hardy or Bergman space of the unit disc, and

$$\lambda_{2j-1} = -\lambda_{2j} = \frac{1}{(j+1)} \quad (j = 1, 2, \dots),$$

then (3) fails for $\varphi(z) = z^2$.

4. Spectral Synthesis: Proof of Main Theorem

We say *spectral synthesis* holds for a bounded sequence $\{z_j\}_1^\infty$ of complex numbers if, whenever $\{\mu_j\}_1^\infty$ is an absolutely summable sequence of complex numbers ($\sum|\mu_j| < \infty$) for which

$$(*) \qquad \sum_{j=1}^{\infty} \mu_j z_j^n = 0 \quad \text{for all integers } n \geq 0,$$

then $\mu_j = 0$ for all j.

An obvious necessary condition for spectral synthesis is that all of the z_j's must be distinct. A sufficient condition is that they should, in addition to being distinct, lie on a Jordan curve [29, Thm. 3].

This concept was first introduced into operator theory in 1952 by John Wermer, who showed that among the normal operators on Hilbert space which have a spanning set of eigenvectors, the ones for which every closed invariant subspace is reducing are exactly those whose sequence of distinct eigenvalues has spectral synthesis [29, Thm. 1].

Subsequently, Brown, Shields, and Zeller gave the following characterization of the spectral synthesis sequences belonging to a Jordan domain [5, Thm. 3, p. 167]:

For a sequence $\{z_j\}_1^\infty$ in a Jordan domain Ω, the following are equivalent:

(a) $\{z_j\}_1^\infty$ *does not have spectral synthesis.*
(b) $\sup_{z\in\Omega}|f(z)| = \sup_j |f(z_j)|$ *for every bounded holomorphic function* f *on* Ω.

In case Ω is the unit disc, Brown, Shields, and Zeller also showed that the following equivalent condition can then be added to the list:

(c) *Almost every boundary point of the unit circle may be approached nontangentially by points of* $\{z_j\}_1^\infty$.

In the last section we encountered the following less restrictive version of spectral synthesis.

4.1. DEFINITION. We call a bounded sequence $\{z_j\}_1^\infty$ of complex numbers an *eventual spectral synthesis sequence* (*an ESS sequence*) if every absolutely summable sequence $\{\mu_j\}_1^\infty$ of complex numbers for which (*) holds must eventually vanish identically (i.e., if $\mu_N=\mu_{N+1}=\cdots=0$ for some N).

According to the work of Section 3, the proof of the Main Theorem will be complete once we prove the following.

4.2. ESS THEOREM. *Each plane domain Ω contains a sequence $\{\lambda_j\}$ of points convergent to a point of the domain, such that the image sequence $\{\varphi(\lambda_j)\}$ is an ESS sequence for every nonconstant holomorphic function φ on Ω.*

We devote the rest of this section to proving this theorem. We assume without further comment that $\{\mu_j\}$ is an absolutely summable sequence of complex numbers. One of the key steps in our analysis is a refinement of the following elementary result.

4.3. LEMMA. *Suppose $\{z_j\}$ is a sequence of complex numbers, arranged in order of decreasing moduli, and suppose that $|z_1|>|z_2|$. If $\{\mu_j\}$ satisfies (*), then $\mu_1=0$.*

Proof. By condition (*) and the fact that $\{|z_j|\}$ is a decreasing sequence, we have, for every nonnegative integer n,

$$|\mu_1|\le\left(\frac{|z_2|}{|z_1|}\right)^n\sum_{j=2}^{\infty}|\mu_j|.$$

Since $|z_1|>|z_2|$, the right side of the last inequality tends to zero as n increases, and this yields the desired result. □

Theorem 2 of [5] provides a more general result, in a somewhat different setting. An immediate consequence of Lemma 4.3 is the next corollary.

4.4. COROLLARY. *Any sequence of complex numbers with strictly decreasing moduli has spectral synthesis.*

Wermer [29, Thm. 4] found a much stronger result than this: *any sequence of distinct complex numbers which contains no infinite subsequence with strictly increasing moduli has spectral synthesis.* His methods figure prominently in our proof of the next result, the "$N=2$" case of which is Lemma 4.3.

4.5. LEMMA. *Suppose $\{z_j\}$ is a complex sequence, arranged in order of decreasing moduli. Suppose that $z_1,z_2,\dots,z_{N-1}$ all have equal moduli, and that $|z_{N-1}|>|z_N|$. Then, whenever (μ_j) satisfies (*), we have*

$$\sum_{j=1}^{N-1} \mu_j z_j^n = \sum_{j=N}^{\infty} \mu_j z_j^n = 0 \quad \textit{for every integer } n \geq 0.$$

Proof. As in the proof of Lemma 4.3, condition $(*)$ and the monotonicity of the sequence of moduli yield, for each $n \geq 0$,

$$\left| \sum_{j=1}^{N-1} \mu_j z_j^n \right| \leq |z_N|^n \sum_{j=N}^{\infty} |\mu_j|,$$

so upon dividing both sides of the inequality above by $|z_1|^n$, letting n tend to ∞, and noting that $|z_N|/|z_1| < 1$, we obtain:

$$\lim_{n \to \infty} \sum_{j=1}^{N-1} \mu_j \left(\frac{z_j}{z_1} \right)^n = 0. \tag{1}$$

Our hypotheses assert that for each $1 \leq j \leq N-1$ there is a real number θ_j such that $z_j/z_1 = e^{2\pi i \theta_j}$. Following Wermer [29, p. 272], we employ a result of Dirichlet [28, pp. 295–296] to provide a strictly increasing infinite sequence $\{t_k\}$ of integers such that, for each $1 \leq j \leq N-1$,

$$\operatorname{dist}(t_k \theta_j, \mathbf{Z}) < 1/k \quad \text{for } k = 1, 2, \ldots, \tag{2}$$

where $\mathbf{Z}$ denotes the collection of integers, and distance is measured in the usual way on the real line. Thus, for each fixed integer $n \geq 0$,

$$\begin{aligned} \sum_{j=1}^{N-1} \mu_j \left(\frac{z_j}{z_1} \right)^n &= \sum_{j=1}^{N-1} \mu_j e^{2\pi i n \theta_j} \\ &= \lim_{k \to \infty} \sum_{j=1}^{N-1} \mu_j e^{2\pi i (n + t_k) \theta_j} \quad &&[\text{by } (2)] \\ &= 0 &&[\text{by } (1)]. \end{aligned}$$

Thus we obtain the first of the desired inequalities:

$$\sum_{j=1}^{N-1} \mu_j z_j^n = 0.$$

The second inequality follows from this one and $(*)$. □

We can now prove the "eventual" version of Corollary 4.4. This result provides a crucial step in our proof of the ESS theorem.

4.6. PROPOSITION. *If $\{z_j\}$ is a sequence of complex numbers with moduli eventually strictly decreasing to zero, then $\{z_j\}$ is an ESS sequence.*

Proof. Without loss of generality, we may assume that $z_j \neq 0$ for all j. Suppose $\{\mu_j\}$ is a fixed sequence that satisfies condition $(*)$. Our goal is to show that $\{\mu_j\}$ eventually vanishes identically. By hypothesis, there is some positive integer M such that the sequence $\{|z_j|\}_{j=M}^{\infty}$ is strictly decreasing to zero. Let $m = \min\{|z_1|, \ldots, |z_{M-1}|\}$ and choose N large enough so that $|z_j| < m$ for $j \geq N$. By rearranging the first $N-1$ terms of the sequence $\{z_j\}$ and making the corresponding rearrangement of $\{\mu_j\}$ so as to preserve (*), we may assume that

$$|z_1| \geq |z_2| \geq \cdots \geq |z_{N-1}| > |z_N| > |z_{N+1}| > \cdots,$$

and that

$$\sum_{j=1}^{\infty} \mu_j z_j^n = 0 \quad \text{for all integers } n \geq 0.$$

Now by applying Lemma 4.5 (perhaps repeatedly), we obtain

$$\text{(1)} \qquad \sum_{j=N}^{\infty} \mu_j z_j^n = 0 \quad \text{for every integer } n \geq 0.$$

Since the sequence $\{z_j\}_{j=N}^{\infty}$ has strictly decreasing moduli, Corollary 4.4 asserts that $\mu_j = 0$ for $j \geq N$. □

The last result shows us that, in order to prove Theorem 4.2, at least for functions φ with $\varphi(0) = 0$ (we suppose for the moment that $0 \in \Omega$), it is enough to choose points $\{\lambda_j\}$ in Ω converging to zero in such a way that, for every such φ, the sequence $\{|\varphi(\lambda_j)|\}$ eventually decreases to zero. The choice of points cannot be made casually. For example, it is not enough for the sequence $\{\lambda_j\}$ merely to have strictly decreasing moduli.

To see this, let $\Omega = U$ and $\varphi(z) = \frac{1}{2}z^2 + \frac{1}{3}z^3$. After sketching the graph of φ over the real axis, the reader can see how to choose points λ_j on the real interval $(-1, 1)$, alternately negative and positive, with strictly decreasing absolute values, so that $\varphi(\lambda_{2j-1}) = \varphi(\lambda_{2j})$ for each j. Thus $\{\varphi(\lambda_j)\}$ is not an ESS sequence.

The goal of the next few paragraphs is to show that this sort of thing does not happen if the sequence $\{\lambda_j\}$ consists of consecutive points lying on a smooth curve through the origin. We begin with a simple geometric fact from calculus.

4.7. CURVE LEMMA. *Suppose that $\gamma: (-1, 1) \to \mathbf{C}$ is continuously differentiable, with $\gamma(0) = 0$ and $\gamma'(0) \neq 0$. Then, for some $0 < \epsilon < 1$, the function $|\gamma|$ is strictly increasing on the interval* $(0, \epsilon)$.

Proof. Since $|\gamma|^2$ is differentiable on $(-1, 1)$, we need only show that its derivative $2\gamma(t) \cdot \gamma'(t)$ (dot product in $\mathbf{R}^2$) is strictly positive on an interval $(0, \epsilon)$. Since $\gamma(0) = 0$, the definition of differentiability asserts that

$$\gamma(t) = t[\gamma'(0) + \delta(t)] \quad (-1 < t < 1),$$

where $\delta: (-1, 1) \to \mathbf{C}$ tends to zero with t. Taking dot products on both sides of the last equation, we have

$$\text{(1)} \qquad \begin{aligned} \gamma(t) \cdot \gamma'(t) &= t[\gamma'(0) \cdot \gamma'(t) + \delta(t) \cdot \gamma'(t)] \\ &= t[|\gamma'(0)|^2 + \gamma'(0) \cdot (\gamma'(t) - \gamma'(0)) + \delta(t) \cdot \gamma'(t)]. \end{aligned}$$

Since γ' is continuous on $(-1, 1)$ and $\delta(t)$ tends to zero with t, we may choose $0 < \epsilon < \frac{1}{2}$ so that, for $|t| < \epsilon$, we have simultaneously

(2) $$|\gamma'(t)-\gamma'(0)|<\frac{1}{4} \quad\text{and}\quad |\delta(t)|\max_{|t|\le 1/2}|\gamma'(t)|<\frac{1}{4}|\gamma'(0)|^2.$$

Thus, for $0<t<\epsilon$, equation (1) implies

$$\begin{aligned}\gamma(t)\cdot\gamma'(t) &\ge t[|\gamma'(0)|^2-|\gamma'(t)-\gamma'(0)||\gamma'(0)|-|\gamma'(t)||\delta(t)|]\\ &\ge t\left[|\gamma'(0)|^2-\frac{1}{4}|\gamma'(0)|^2-\frac{1}{4}|\gamma'(0)|^2\right] \qquad \text{[by (2)]}\\ &=\frac{t}{2}|\gamma'(0)|^2>0,\end{aligned}$$

as desired. Thus $|\gamma|^2$ is strictly increasing on $(0,\epsilon)$ and hence so is $|\gamma|$. □

Our goal in this section is to prove Theorem 4.2. The following consequence of Lemma 4.7 does this, at least for φ's that vanish somewhere on Ω, and therefore *almost* completes the proof of the Main Theorem.

4.8. COROLLARY. *Suppose that $\gamma:(-1,1)\to\Omega$ is continuously differentiable, with $\gamma'(0)\neq 0$. Let $\{t_j\}$ be a sequence in the open unit interval that decreases strictly to zero, and define $\lambda_j=\gamma(t_j)$ $(j=1,2,\ldots)$. Then for every holomorphic function φ on Ω, with $\varphi(\gamma(0))=0$, the sequence of moduli $\{|\varphi(\lambda_j)|\}$ eventually strictly decreases to zero.*

Proof. Let $\alpha=\gamma(0)$. Since $\varphi(\alpha)=0$ there is a nonnegative integer m for which φ can be factored as

$$\varphi(z)=(z-\alpha)^m\psi(z)\quad (z\in\Omega),$$

where ψ is holomorphic on Ω and vanishes at α, but $\psi'(\alpha)\neq 0$.

The conditions imposed on ψ and γ ensure that both $\gamma-\alpha$ and $\psi\circ\gamma$ satisfy the hypotheses of Lemma 4.7, so their moduli both increase strictly on some interval $(0,\epsilon)$. Thus the same is true of the modulus of $\varphi\circ\gamma$, which immediately yields the desired property of the image sequence $\{\varphi(\lambda_j)\}$. □

The restriction that φ vanish at some point of Ω is easily disposed of.

4.9. LEMMA. *If $\{z_j\}$ is an ESS sequence, then so is the sequence $\{z_j+\alpha\}$ for every complex number α.*

Proof. Suppose $\{\mu_j\}$ is an absolutely summable sequence for which

(1) $$\sum_{j=1}^{\infty}\mu_j(z_j+\alpha)^n=0 \quad\text{for every integer } n\ge 0.$$

We must show that all but finitely many of the terms μ_j vanish. Upon expanding the summand in (1) by the binomial theorem and interchanging the order of summation, we obtain

$$0=\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\left(\sum_{j=1}^{\infty}\mu_j z_j^n\right)\quad (n=0,1,2,\ldots).$$

We have here a lower triangular system of equations, which can be successively solved for the inner sum. The result is

$$\sum_{j=1}^{\infty} \mu_j z_j^n = 0 \quad (n=0,1,2,\ldots).$$

But we are assuming that $\{z_j\}$ is an ESS sequence, and hence all but finitely many of the terms μ_j are zero, as desired. □

Putting the last two results together yields the next corollary.

4.10. COROLLARY. *Suppose the curve γ and the sequence $\{\lambda_j\}$ are as in the hypotheses of Corollary 4.8. Then, for any nonconstant holomorphic function φ on Ω, the image sequence $\{\varphi(\lambda_j)\}$ is an ESS sequence.*

Proof. By Corollaries 4.6 and 4.8, the sequence $\{\varphi(\lambda_j)-\varphi(\gamma(0))\}$ has the desired property; hence, by Lemma 4.9, so does $\{\varphi(\lambda_j)\}$. □

The proof of Theorem 4.2, and therefore of the Main Theorem, is now complete. For the benefit of the reader we summarize the findings of the last two sections.

4.11. MAIN THEOREM REVISITED. *Suppose that $\gamma:(-1,1)\to\Omega$ is continuously differentiable, with $\gamma'(0)\neq 0$. For a sequence $\{t_j\}$ in the open unit interval that decreases strictly to zero, define $\lambda_j=\gamma(t_j)$ $(j=1,2,\ldots)$. Then, for any absolutely summable sequence $\{b_j\}$ of complex numbers with infinitely many nonzero terms, the function f defined by the series*

$$f=\sum_{j=1}^{\infty} b_j k_{\lambda_j}$$

lies in H and is a cyclic vector for every operator M_φ^*, where φ is a nonconstant multiplier of H.

4.12. REMARKS. (a) Since the points $\{\lambda_j\}$ cluster in Ω, the corresponding reproducing kernels span a dense subspace of H. Thus the explicit form given for the "universally cyclic" vectors f described above shows that the collection of such vectors is, in fact, dense in H. The requirement that infinitely many of the coefficients b_j should not vanish prevents the f's described above from forming a linear subspace, but we will see in the next section that *there is a dense linear subspace of H that consists (except for zero) of common cyclic vectors for the adjoints of the nonscalar multiplication operators.*

(b) Nikolskii has observed that the curve lemma can be avoided (albeit with some loss in generality) if one requires that the sequence $\{\lambda_j\}$ converge *geometrically* to a point $a\in\Omega$:

$$|\lambda_{j+1}-a|/|\lambda_j-a|\le q<1 \quad \text{for all } j.$$

For in this case, if φ is a holomorphic function on Ω, then a simple power series argument shows that

$$|\varphi(\lambda_{j+1})-\varphi(a)|/|\varphi(\lambda_j)-\varphi(a)|\le (q+1)/2<1$$

for all sufficiently large j. Thus $\{\varphi(\lambda_j)\}$ has eventually strictly decreasing moduli and so is an ESS sequence by Proposition 4.6.

4.13. MAIN THEOREM IN $\mathbf{C}^N$. The Main Theorem generalizes verbatim to arbitrary domains in $\mathbf{C}^N$, where $N>1$. Here we indicate the changes needed to make the proof work in this new setting.

Let Ω denote such a domain, and (as in the previous work) let H denote a Hilbert space of functions holomorphic on Ω such that, for each point $\lambda \in \Omega$, the linear functional of evaluation at λ is continuous on H.

The discussion in Section 1 of the reproducing kernels and multipliers in such spaces goes through, practically unchanged, to this higher-dimensional setting. The only argument requiring extra care is the proof of Proposition 1.2, the result showing each multiplier of H to be a bounded holomorphic function on Ω. The argument given shows that each multiplier is holomorphic off the zero set of some nontrivial holomorphic function on Ω. The difference here is that in higher dimensions zero sets are *never* discrete, so zeros cannot be removed one at a time. But Riemann's original theorem, applied to the restriction of our multiplier to the intersection of each complex line with Ω, shows that the function under scrutiny is holomorphic on every such set, and therefore holomorphic on all of Ω (see [21, Chap. 4, Cor., p. 62]). So Proposition 1.2 continues to hold in higher dimensions.

Let us call a subset S of Ω a *set of uniqueness for* $H(\Omega)$ if the only holomorphic function vanishing on S is the zero function. In dimension 1, the sets of uniqueness are precisely the subsets of Ω having a limit point in Ω, but as we observed in the last paragraph, in higher dimensions the situation is considerably more complicated.

The essence of our method for constructing common cyclic vectors for adjoint multipliers can be summarized, in any dimension, as follows.

PROPOSITION. *Suppose there exists a sequence $\{\lambda_j\}$ in Ω such that:*

(i) *$\{\lambda_j\}$ is a set of uniqueness for $H(\Omega)$; and*

(ii) *for every nonconstant function φ holomorphic on Ω, the image sequence $\{\varphi(\lambda_j)\}$ is an ESS sequence.*

Then, for each absolutely summable sequence $\{b_j\}$ of nonzero complex numbers, the series $\sum b_j k_{\lambda_j}$ converges to a function $f \in H$ that is a cyclic vector for every operator M_φ^*, where φ is a nonconstant multiplier of H.

Thus the extension of the Main Theorem from one to many complex dimensions is performed by the following result, whose proof was provided for us by Wade Ramey.

THEOREM. *Suppose Ω is a domain in $\mathbf{C}^N$ and $\alpha \in \Omega$. Then there exists a sequence $\{\lambda_j\}$ in Ω such that:*

(a) $\lambda_j \to \alpha$;

(b) *$\{\lambda_j\}$ is a set of uniqueness for $H(\Omega)$; and*

(c) *for every nonconstant φ holomorphic on Ω, the image sequence $\{\varphi(\lambda_j)\}$ is an ESS sequence.*

Proof (W. Ramey). For simplicity, let us initially suppose that α is the point with all coordinates "1", and Ω contains the closure of the *unit polydisc* U^N. Thus the "distinguished boundary"

$$\mathbf{T}^N = \{(\omega_1, \omega_2, \ldots, \omega_N) : |\omega_j| = 1 \text{ for } 1 \le j \le N\}$$

of U^N also lies in Ω. We note that $\mathbf{T}^N$ is a "maximum modulus set" for functions holomorphic in a neighborhood of the closure of U^N: The modulus of each such function attains its supremum over U^N at some point of $\mathbf{T}^N$ [20, Thm. 2.1.3, p. 18]. In particular, $\mathbf{T}^N$ is a set of uniqueness for functions that are holomorphic in a neighborhood of the closure of U^N.

Fix real numbers $\sigma_1, \sigma_2, \ldots, \sigma_N$, linearly independent over the rationals, and define a mapping $\gamma : \mathbf{R} \to \mathbf{T}^N$ by

$$\gamma(t) = (e^{i\sigma_1 t}, e^{i\sigma_2 t}, \ldots, e^{i\sigma_N t}) \quad (t \in \mathbf{R}).$$

Because of our choice of the parameters σ, the curve $\gamma(\mathbf{R})$ is dense in T^N and is therefore a set of uniqueness for functions holomorphic on Ω. More is true:

Every subset of $\gamma(\mathbf{R})$ *with a limit point is a set of uniqueness for* $H(\Omega)$.

For suppose $\{t_j\}$ is a convergent sequence of distinct real numbers. If g is holomorphic on Ω and vanishes at each point $\gamma(t_j)$, then the composition $G = g \circ \gamma$ is real-analytic on $\mathbf{R}$ and vanishes at each point t_j. Thus $G \equiv 0$ on $\mathbf{R}$, so $g \equiv 0$ on $\gamma(\mathbf{R})$ and therefore on $\mathbf{T}^N$. Thus $g \equiv 0$ on Ω, since, as we pointed out above, $\mathbf{T}^N$ is a set of uniqueness for functions holomorphic on Ω.

Now let $\lambda_j = \gamma(1/j)$ for $j = 1, 2, \ldots$. Thus $\{\lambda_j\}$ is a sequence in $\gamma(\mathbf{R})$ that converges to α. By the last paragraph, $\{\lambda_j\}$ is a set of uniqueness for $H(\Omega)$; we claim it has the desired spectral synthesis property.

To see this, suppose φ is holomorphic and nonconstant on Ω. Then the composition $\varphi \circ \gamma$ is real-analytic on $\mathbf{R}$, so there exists $\epsilon > 0$ and an integer $M \ge 0$ such that

$$\varphi \circ \gamma(t) - \varphi \circ \gamma(0) = t^M \psi(t) \quad (|t| \le \epsilon),$$

where ψ is a complex valued real-analytic function on $(-\epsilon, \epsilon)$, with $\psi'(0) \neq 0$. By the curve lemma, for some $0 < \epsilon' < \epsilon$, the modulus of ψ is strictly increasing on the interval $[0, \epsilon']$, so the same is true of $\varphi \circ \gamma$. Thus the sequence $\{\varphi(\lambda_j) - \varphi(\alpha)\}$ has moduli that are eventually strictly decreasing, so $\{\varphi(\lambda_j)\}$ is an ESS sequence by Propositions 4.6 and 4.9. This completes the proof, under the special assumptions that Ω contains the unit polydisc and α lies on the distinguished boundary of that polydisc.

For the general case, one need only repeat the argument with the unit polydisc replaced by an appropriate translated dilate. We leave the details to the reader. □

5. Final Remarks

We collect here a few complements to the work of the previous sections, and comment on some related work in the literature.

Cyclic vector manifolds. In [11], Godefroy and Shapiro show that nonscalar operators that commute with variants of the backward shift have dense invariant manifolds of cyclic vectors. Their idea works in the present context, and yields an improvement of the Main Theorem.

5.1. THEOREM. *There is a dense linear manifold of H, invariant under the adjoint of every multiplication operator, each nonzero element of which is cyclic for the adjoint of every nonscalar multiplication operator.*

Proof. Let f be one of the vectors promised by the Main Theorem. We claim that

$$L=\{M_\varphi^* f : \varphi\in \mathfrak{M}(H)\}$$

has the desired properties. Clearly L is a linear submanifold of H. To see that L has the desired invariance, choose multipliers φ and ψ of H. Then $\varphi\psi$ is also a multiplier, and

$$M_\psi^*(M_\varphi^* f)=(M_\varphi M_\psi)^* f=M_{\varphi\psi}^* f\in L,$$

so L is M_ψ^*-invariant. Moreover, L is dense in H because it contains orbits of the form $\{(M_\varphi^*)^n f\}$, and such orbits have dense linear span in H provided the multiplier φ is nonconstant.

It remains to show that if $\psi\in\mathfrak{M}(H)$ is not constant then each nonzero element of L is cyclic for M_ψ^*. Fix $0\neq\varphi\in\mathfrak{M}(H)$, so $M_\varphi^* f$ is a typical nonzero element of L. Since M_φ^* and M_ψ^* commute, we have

$$\operatorname{span}\{(M_\psi^*)^n M_\varphi^* f\}_{n=0}^\infty=M_\varphi^*[\operatorname{span}\{(M_\psi^*)^n f\}_{n=0}^\infty].$$

Since f is cyclic for M_ψ^*, the set in brackets on the right is dense in H. Since M_φ is one-to-one ($\varphi\neq 0$), its adjoint has dense range, so the left side of the equation above emerges as the image of a dense set under an operator with dense range. It is therefore dense, and the proof is complete. □

Operators on the space of entire functions that commute with translation. Our method applies as well to the space $H(\mathbf{C}^N)$ of entire functions on $\mathbf{C}^N$, endowed with the topology of uniform convergence on compact sets, where it produces common cyclic vectors (even manifolds of them, in the sense of Theorem 5.1) for the class of nonscalar continuous linear operators that commute with translations.

The idea behind the proof is that each of our operators can be written in the form $L=\Phi(D_1,D_2,\dots,D_N)$, where Φ is an entire function of exponential type and $D_1,D_2,\dots,D_N$ are the usual partial differentiation operators (see, e.g., [11, §5]. For $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_N)\in\mathbf{C}^N$, the function e_α defined by

$$e_\alpha(z)=\exp(\alpha_1 z_1+\alpha_2 z_2+\cdots+\alpha_N z_N)\quad(\alpha\in\mathbf{C}^N)$$

is an eigenvector for D_j with eigenvalue α_j, and is therefore also an eigenvector for the above operator $L=\Phi(D)$, this time with eigenvalue $\Phi(\alpha)$. The proof of our Main Theorem can now be repeated, with the eigenvectors $\{e_\alpha\}$ replacing the Hilbert space reproducing kernels, and with the orthogonality argument replaced by an application of the Hahn–Banach theorem. This latter step requires the fact that every continuous linear functional on $H(\mathbf{C}^N)$ can be represented by integration against a Borel measure on $\mathbf{C}^N$ of compact support. We leave the details to the reader.

Very strong cyclicity theorems were proved for the operators of differentiation and translation on $H(\mathbf{C})$ by Birkhoff [3] and MacLane [15]. These results were recently generalized by Godefroy and Shapiro [11] to operators of the sort discussed above. The result indicated above generalizes work of Chan [7], who found common cyclic vectors for the class of positive order partial differential operators with constant coefficients.

Comparison with work of Clancey and Rogers. In 1978, Clancey and Rogers studied cyclic vectors for hyponormal and co-hyponormal operators [8]. An important result in their paper was the following.

5.2. THEOREM [8, Thm. 3]. *If an operator T on Hilbert space has the following property:*

(CR) $E(T) := \operatorname{span}\{\ker(T-\lambda I) : \lambda \in \mathbf{C}, T-\lambda I$ *surjective*$\}$ *is dense,*

then T has a dense set of cyclic vectors.

Clancey and Rogers also pointed out a connection between their result and spectral synthesis [8, §4]. Our Main Theorem provides cyclic vectors for the adjoints of multiplication operators in a very general setting (in fact, it provides common cyclic vectors). Although the Clancey–Rogers theorem shows that adjoints of some multiplication operators have cyclic vectors, we will now show that it does not apply to all of them.

In order to state our result, we need to introduce the *little Bloch space:* This is the collection $\mathcal{B}_0$ of functions f holomorphic on the unit disc U for which

$$|f'(z)| = o\left(\frac{1}{1-|z|}\right) \quad \text{as } |z| \to 1^-.$$

Every bounded holomorphic function on U satisfies the corresponding "big oh" condition (the collection of holomorphic functions satisfying this condition is the *big Bloch space,* or just the *Bloch space*). Although it is clear that every finite Blaschke product belongs to $\mathcal{B}_0$, the existence of infinite Blaschke products in $\mathcal{B}_0$ was only recently observed by Sarason [22, p. 337]. Since then, Bishop [4] has given a more constructive proof of the existence of such Blaschke products.

We can now present the promised result, which shows that the existence of a dense set of cyclic vectors for our adjoint multiplication operators does not, in general, follow from Theorem 5.2.

5.3. THEOREM. *If φ is a Blaschke product in the little Bloch space, and M_φ is viewed as an operator on the Bergman space $L_a^2(U)$, then $E(M_\varphi^*) = \{0\}$.*

Proof. Suppose initially that φ is just a bounded holomorphic function on U. Then, for each complex number λ, since $M_\varphi^* - \bar{\lambda} I = M_{\varphi-\lambda}^*$ we have

$$\ker(M_\varphi^* - \bar{\lambda} I) = \ker(M_{\varphi-\lambda}^*) = [(\varphi-\lambda)L_a^2(U)]^\perp.$$

Since an operator is surjective if and only if its adjoint is bounded below,

$$E(M_\varphi^*) = \operatorname{span}\{[(\varphi-\lambda)L_a^2(U)]^\perp : M_{\varphi-\lambda} \text{ bounded below on } L_a^2(U)\}. \tag{1}$$

Regarding boundedness below of multiplication operators on the Bergman space, Sheldon Axler has pointed out to us the following useful fact:

If $\varphi \in \mathcal{B}_0$, then M_φ is not bounded below on $L_a^2(U)$.

Axler's proof is as follows. In [2], he proves that the self-commutator

$$S_\varphi = M_\varphi^* M_\varphi - M_\varphi M_\varphi^*$$

is compact on $L_a^2 = L_a^2(U)$ if (and only if) $\varphi \in \mathcal{B}_0$. Now the L_a^2-reproducing kernel for a point $\lambda \in U$ is:

$$k_\lambda(z) = \frac{1}{\pi(1-\bar{\lambda} z)^2} \quad (z \in U).$$

As λ tends to ∂U, the normalizations f_λ defined by

$$f_\lambda(z) = \frac{k_\lambda(z)}{\|k_\lambda\|} = \frac{1}{\sqrt{\pi}} \frac{(1-|\lambda|^2)}{(1-\bar{\lambda} z)^2} \quad (z \in U)$$

clearly converge to zero uniformly on compact subsets of U, and therefore, by their norm boundedness and the continuity of point evaluations, converge to zero weakly in L_a^2.

Now suppose φ is an inner function in $\mathcal{B}_0$, but not a finite Blaschke product. By the compactness of the self-commutator S_φ and the above-mentioned weak convergence, $\|S_\varphi f_\lambda\| \to 0$ as $|\lambda| \to 1^-$; hence

$$0 = \lim_{|\lambda| \to 1^-} \langle S_\varphi f_\lambda, f_\lambda \rangle = \lim_{|\lambda| \to 1^-} (\|\varphi f_\lambda\|^2 - |\varphi(\lambda)|^2 \|f_\lambda\|^2), \tag{2}$$

where the last equality arises from computation of the inner product, using Proposition 1.3. Since φ is not a finite Blaschke product, there is a sequence $\{\lambda_j\}$ in U, tending to the boundary, such that $\varphi(\lambda_j) \to 0$. Upon substituting λ_j for λ in (2) above we obtain

$$\lim_{j \to \infty} \|M_\varphi f_{\lambda_j}\| = 0.$$

Since each f_λ has norm 1, this shows that M_φ is not bounded below, as desired. □

We can now complete the proof of Theorem 5.3. As before, fix φ an inner function in $\mathcal{B}_0$, not a finite Blaschke product. Our goal is to show that $E(M_\varphi^*) = \{0\}$. For each $\lambda \in U$, the function

$$\varphi_\lambda = \frac{\varphi - \lambda}{1 - \bar{\lambda}\varphi}$$

is again in $\mathcal{B}_0$ and not a finite Blaschke product; so, by the argument above, M_{φ_λ} is not bounded below on L_a^2 and hence neither is

$$M_{\varphi-\lambda}=M_{1-\bar{\lambda}\varphi}M_{\varphi_\lambda}.$$

Thus the complex numbers that figure in the right side of the expression (1) for $E(M_\varphi^*)$ must lie outside the open unit disc. If they lie outside the *closed* unit disc, then $M_{\varphi-\lambda}$ is invertible and hence $(\varphi-\lambda)L_a^2(U)=L_a^2(U)$. So, in fact, only points λ on the unit circle can contribute to the right side of (1). But even these contribute nothing. If $|\lambda|=1$, then λ belongs to the boundary of the spectrum of M_φ and hence to the approximate point spectrum of M_φ. Thus $M_{\varphi-\lambda}$ cannot be bounded below for any λ on the unit circle.

Summarizing the work of the last few paragraphs, we have

$$\begin{aligned} E(M_\varphi^*) &= \operatorname{span}\{[(\varphi-\lambda)L_a^2(U)]^\perp : M_{\varphi-\lambda} \text{ bounded below on } L_a^2(U)\} \\ &\subset \operatorname{span}\{[(\varphi-\lambda)L_a^2(U)]^\perp : |\lambda|\ge 1\} \\ &= \{0\}. \end{aligned}$$ □

We remark that MacDonald and Sundberg [16, Cor. 23, p. 610] have shown that the multiplication operator induced on $L_a^2(U)$ by an inner function is bounded below on $L_a^2(U)$ if and only if the function is a finite product of interpolating Blaschke products. The result of Axler mentioned above shows that no nontrivial inner function in the little Bloch space is a finite product of interpolating Blaschke products.

Additional information relating to Theorem 5.2 can be found in recent work of Nikolskii and Vasyunin ([17], [18]).

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