RESEARCH STATEMENT

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My research interests, broadly stated, are the theory of partial differential equations and the calculus of variations. In particular, I am interested in the qualitative theory of elliptic and parabolic equations and regularity property of minimizer of a certain nonlinear elliptic systems. Here is a brief description of the results I have obtained so far along with future plans.

1. RESEARCH SUMMARY

1.1. *Partial regularity result and Hausdorff dimension estimate*

Consider the integral functionals of the type

$$I[u] := \int_\Omega F(x, u(x), Du(x))\, dx$$

defined for maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of the Sobolev class $W^{1,p}(\Omega)$, $p > 1$, $N \geq 2$. Extensive important results obtained related to the regularity of the minimizer or solution to Euler-Lagrange equation when we impose some strict convex condition on F, such as strong ellipticity, Legendre-Hadamard, or scalar case (N=1) [G, Gi].

But a less strict condition, the quasiconvex condition, that is,

$$\int_O F(A)dy \leq \int_O F(A + D\phi)dy$$

for all smooth, bounded, open domain O and $A \in M^{n\times N}$, all $\phi \in C_0^\infty(O)$, is also important since Morrey's paper [Mo52] and [AF] where they showed that quasiconvexity condition is equivalent to weakly sequentially lower semicontinuous of I.

Evans showed the partial regularity of the minimizer for the uniformly strictly quasiconvex case, that is,

$$\int_O F(A) + \gamma|D\phi|^2 dy \leq \int_O F(A + D\phi)dy$$

for some $\gamma > 0$ and all smooth, bounded open domain O, all matrices $A \in M^{n\times N}$ and all $\phi \in C_0^\infty(O)$ with a some proper growth condition on F. Refer to [Ev87] for more details. Here the partial regularity means the following:

Theorem 1.1. *Let u be a minimizer of I. Then there exists an open subset Ω_0 of Ω such that*

$$|\Omega \setminus \Omega_0| = 0$$

and

$$Du \in C^\alpha(\Omega_0; M^{n\times N})$$

for some $\alpha > 0$.

Recently in a series of papers, G. Mingione, et al show that the singular set $\Omega_1 := \Omega \setminus \Omega_1$ has controllable Hausdorff dimension, namely $\mathcal{H}(\Omega_1) < n$ with a strict convex condition. For the quasi-convex case, J. Kristensen and G. Mingione also obtained the similar result under the condition that the minimizer u is Lipschitz continuous. See [Mi, KM] for more details.

I am interested in generalizing their result in the following polyconvex functional:

$$I[u] := \int_\Omega |Du|^2 + \det{}^2 Du.$$

Its Caccioppoli inequality and partial regularity have been obtained by N. Fusco and J. Hutchinson [FH94]. The main difficulties comes from the facts that $\det^2 Du$ has growth of $|Du|^4$ and $\det$ involve the nonlinearity.

In our joint work with X. Yan, following [FH91], we are able to derive the partial regularity of the critical points of this functional along with Hausdorff dimension estimate under the assumption that the solution u has small Lipschitz norm. It will be interesting to compare this results with the counterexample by [MS].

1.2. *Hölder regularity of solutions to second order elliptic equations in non-smooth domains*

J. Michael [Mc77, Mc81] investigated the Dirichlet problem

$$Lu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where f is allowed to be unbounded near $\partial\Omega$ (see (3)), and L is a uniformly elliptic operator of nondivergence form:

$$Lu := -\sum_{i,j=1}^{n} a_{ij} D_{ij} u = -(aD, Du). \tag{ND}$$

The uniform ellipticity means that there exists a constant $\nu \in (0, 1]$ such that, for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, we have

$$\nu|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j = (a\xi, \xi), \qquad \max_{|\xi|\le 1} |a\xi| \le \nu^{-1}. \tag{2}$$

Using Miller's barrier technique [Ml67], Michael developed a general Schauder type existence theory, which is based on interior estimates only. The key element in his approach is the following inequality for solutions to the problem (1):

$$\sup_\Omega d^{-\gamma}|u| \le N \sup_\Omega d^{2-\gamma}|f|, \qquad N = N(n, \nu, \gamma), \tag{3}$$

where $d = d(x) := \mathrm{dist}(x, \partial\Omega)$, $0 < \gamma < \gamma_0$, γ_0 is a constant depending only on n, ν, and the characteristics of exterior cones for domain Ω. At about the same time, D. Gilbarg and L. Hörmander [GH] also used Miller's barriers in their theory of intermediate Schauder estimates (see also Section 6.5, [GT]).

Unfortunately, these approaches rely heavily on the explicit form of barrier functions, therefore they are not readily extended to the divergence case. In our joint work with M. Safonov [CS], we presented an alternative method to obtain the inequality (3), which is equally applicable to both nondivergence (ND) and divergence (D) form of elliptic operators L, where

$$Lu := -\sum_{i,j=1}^{n} D_i(a_{ij} D_j u) = -(D, aDu). \tag{D}$$

Moreover, it works for more general bounded domains Ω which satisfy the following: *there exists a constant* $\theta_0 > 0$ *such that, for any* $y_0 \in \partial\Omega$ *and* $r > 0$,

$$|B_r(y_0) \setminus \Omega| \geq \theta_0 |B_r|.$$

Also, we derived Hölder regularity as follows:

$$|u|_{0,\gamma;\Omega} \leq N \sup_{\Omega} d^{2-\gamma}|f|$$

for solutions to the problem (1) with a divergence (D) or nondivergenc (ND) form of L in a nonsmooth domain described above. One of the key ingredients for our proof is the following Growth lemma (see also [La, KS, K, FeS]).

Lemma 1.2 (Growth Lemma)**.** *Let* $x_0 \in \mathbb{R}^n$ *and* $r > 0$ *be such that*

$$|B_r \setminus \Omega| \geq \theta\, |B_r|\,, \quad \theta > 0,$$

where $B_r := B_r(x_0)$. *Then, for any function* $u \in C^2_{loc}(\Omega) \cap C(\overline{\Omega})$ *satisfying* $u > 0$, $Lu \leq 0$ *in* Ω, *and* $u = 0$ *on* $(\partial\Omega) \cap B_{4r}$, *we have*

$$\sup_{\Omega \cap B_r} u \leq \beta \cdot \sup_{\Omega \cap B_{4r}} u,$$

where $\beta = \beta\,(n, \nu, \theta) \in (0, 1)$.

Using the parabolic cylinder $C_r(X) := B_r(x) \times (t - r^2, t), X = (x,t)$, our methods are extended to the corresponding parabolic case.

1.3. *Two sided global estimates of the Green's function of parabolic equations*

The following bounds on the fundamental solution Γ for the uniformly parabolic divergence form operator L are well known from the result of D. Aronson [Ar]. There exists positive constants c_1 and c_2 such that

$$\frac{1}{c_1}\Gamma_{c_2}(X,Y) \leq \Gamma(X,Y) \leq c_1 \Gamma_{\frac{1}{c_2}}(X,Y), \quad t > s, \tag{4}$$

where $\Gamma_{c_2}(X,Y) := \frac{1}{(t-s)^{n/2}} e^{-c_2 \frac{|x-y|^2}{t-s}}$, $X = (x,t), Y = (y,s)$. Here,

$$Lu := D_t u - (D, aDu) = \sum_{i,j=1}^{n} D_t u(x,t) - D_j(a_{ij}(x,t) D_i u(x,t)),$$

and the uniform parabolicity means that, similar to (2), there exists a constant $\nu \in (0,1]$ such that, for all $X = (x,t) \in Q$ and $\xi \in \mathbb{R}^n$, we have

$$\nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j = (a\xi,\xi), \qquad \max_{|\xi| \leq 1} |a\xi| \leq \nu^{-1}. \tag{5}$$

D. Aronson used the Harnack inequality in his proof, which is due to J. Moser [Mo]. Using some ideas of J. Nash [Na], E. Fabes and D. Stroock [FSt] derived the inequalities (4) directly, and they apply them to obtain Moser's Harnack inequality. Generalizing the bounds (4) to the Green's function G of the operator L in a bounded smooth cylinder $Q := D \times (0,T)$, we expect the following: there exist fixed constants c_1 and c_2 depending on the original quantities, for $X = (x, t), Y = (y, s) \in Q$, $t > s$, such that

$$m(X, Y)\frac{1}{c_1}\Gamma_{c_2}(X, Y) \le G(X, Y) \le m(X, Y)c_1\Gamma_{\frac{1}{c_2}}(X, Y), \tag{6}$$

$$\text{where } m(X, Y) := \left(1 \wedge \frac{d(x)}{\sqrt{t-s}}\right) \cdot \left(1 \wedge \frac{d(y)}{\sqrt{t-s}}\right),$$

$d(x) := \mathrm{dist}(x, \partial D)$, $a \wedge b := \min[a, b]$. Observe that the inequalities (6) imply the bounds on the Poisson kernel $P = \frac{\partial}{\partial \nu} G$. With time-independent coefficients a_{ij}, E. Davies and B. Simon [Da, DS] proved the upper bound even with a sharp constant c_2, and Q. Zhang [Zh02] derived the lower bound using [FSt, FGS]. Utilizing the ideas of M. Grüter and K. Widman [GW], R. Finn and D. Gilbarg [FG], I extended the inequalities (6) to time-dependent case with Dini-continuous coefficients a_{ij} and $C^{1,\alpha}$ domain D [Ch].

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