MATRIX LIE GROUPS AND LIE GROUPS

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I. MATRIX LIE GROUPS

Definition: A **matrix Lie group** is a closed subgroup G_M of $GL(n, \mathbb{C})$

Thus if $\{\underset{\sim}{A}_m\}_{m=1}^{\infty}$ is any sequence of matrices in G_M, and $\underset{\sim}{A}_m \to \underset{\sim}{A}$ for some $\underset{\sim}{A} \in M_n(\mathbb{C})$, then either $\underset{\sim}{A} \in G_M$ or $\underset{\sim}{A}$ is not invertible.

Example of a Group that is Not a Matrix Lie Group

Let $G^n_{\mathbb{Q}} = \{\underset{\sim}{A} \in GL(n, \mathbb{C}) : \underset{\sim}{A} = [a_{ij}]^{n,n}_{i,j=1},$ where $a_{ij} \in \mathbb{Q},\ \forall i, j\}$.

Then there exists $\{\underset{\sim}{A}_m\}_{m=1}^{\infty} \subseteq G^n_{\mathbb{Q}}$ such that $\underset{\sim}{A}_m \to \pi\underset{\sim}{\delta} \notin G_M$, but $\pi\underset{\sim}{\delta}$ **is** invertible.

Thus $G^n_{\mathbb{Q}}$ is **not** a matrix Lie group.

Examples of Matrix Lie Groups

$O(n), U(n), Sp(n)$, etc.

Another example is the **Heisenberg group** H described below:

$$\text{Let } A = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

Note: If $\underset{\sim}{B}, \underset{\sim}{C} \in A$, then $\underset{\sim}{B}\underset{\sim}{C} \in A$

$$\text{If } \underset{\sim}{B} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } \underset{\sim}{B}^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \in A.$$

Then $(A, \cdot)$ is a subgroup of $GL(3, \mathbb{R})$ and if $\{\underset{\sim}{B}_m\}_{m=1}^{\infty} \subseteq A$ with $\underset{\sim}{B}_m \to \underset{\sim}{B}$, then $\underset{\sim}{B} \in A$, so $(A, \cdot)$ is a matrix Lie group, called the Heisenberg group H.

Facts about the Heisenberg group H

1. $Z(H) = \left\{ \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : b \in \mathbb{R} \right\}$ [Exercise]

2. Let $\mathfrak{h} = \left\{ \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$. Then $\mathfrak{h}$ is the Lie algebra of H. [Exercise]

3. Let $\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\underset{\sim}{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\underset{\sim}{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

   Then $\{\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}\}$ is a basis for $\mathfrak{h}$.

4. $\underset{\sim}{B}^2 = \underset{\sim}{0}$ so $e^{t\underset{\sim}{B}} = \underset{\sim}{\delta} + t\underset{\sim}{B} = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   Thus $Z(H) = \{e^{t\underset{\sim}{B}} : t \in \mathbb{R}\}$.

5. $[\underset{\sim}{A}, \underset{\sim}{C}] = \underset{\sim}{B}$ and $[\underset{\sim}{A}, \underset{\sim}{B}] = [\underset{\sim}{C}, \underset{\sim}{B}] = \underset{\sim}{0}$ [Exercise]

II. LIE GROUPS

Definition: G_L is a $(C^\infty)-$ **Lie group** if $G_L = (X, \mathcal{T}, |\mathcal{A}|, \cdot)$ where

1. $(X, \mathcal{T}, |\mathcal{A}|)$ is a C^∞-manifold

2. $(X, \cdot)$ is a group

3. $\cdot : X \times X \to X$ is a smooth function with respect to the smooth product structure on $X \times X$ and the smooth structure on X (determined by $|\mathcal{A}|$)

4. Letting $\mathscr{inv} : X \to X$ be defined by $\mathscr{inv}(x) = x^{-1}$, $\mathscr{inv}$ is a smooth function (with respect to $|\mathcal{A}|$)

The Lie Group G

Let $X = \mathbb{R} \times \mathbb{R} \times S^1_{\mathbb{C}} = \left\{(x, y, u) : x, y \in \mathbb{R} \text{ and } u \in S^1_{\mathbb{C}} = \{z \in \mathbb{C} : |z| = 1\}\right\}$.

Let $\mathcal{T}_{S^1_{\mathbb{C}}}$ be the subspace topology on $S^1_{\mathbb{C}}$ induced from $\mathbb{C}$.

Let $\mathcal{T}_X$ be the standard product topology on X induced from $\mathcal{T}_{\mathbb{R}}$, the standard metric topology, and $\mathcal{T}_{S^1_{\mathbb{C}}}$.

Note: $\mathcal{T}_X$ is second countable and Hausdorff

Let $U_1 = \{e^{i\theta} : \theta_1 \in (0, 2\pi)\}$ and $U_2 = \{e^{i\theta} : \theta_2 \in (-\pi, \pi)\}$.

Let $V_1 = (0, 2\pi)$ and $V_2 = (-\pi, \pi)$.

Let $\varphi_1 : U_1 \to V_1$ be defined by $z \mapsto \{\text{unique } \theta_1 \in (0, 2\pi) \text{ such that } z = e^{i\theta_1}\}$.

Let $\varphi_2 : U_2 \to V_2$ be defined by $w \mapsto \{\text{unique } \theta_2 \in (-\pi, \pi) \text{ such that } w = e^{i\theta_2}\}$.

Let $\mathcal{A}_{S^1_{\mathbb{C}}} = \{(U_1, V_1, \varphi_1), (U_2, V_2, \varphi_2)\}$

Then $(S^1_{\mathbb{C}}, \mathcal{T}_{S^1_{\mathbb{C}}}, |\mathcal{A}_{S^1_{\mathbb{C}}}|)$ is a C^∞-manifold. [Exercise]

Via the standard product construction, with product atlas $\mathcal{A}_X$, we have that $(X, \mathcal{T}_X, |\mathcal{A}_X|)$ is a C^∞-manifold. [Exercise]

Note: If $z \in X$, then in each coordinate chart, $z = (x, y, e^{i\theta})$ for an appropriate choice of θ.

Define: $\cdot : X \times X \to X$ locally (i.e. in each coordinate chart) as

$$(x_1, y_1, u_1) \cdot (x_2, y_2, u_2) = (x_1 + x_2, y_1 + y_2, e^{ix_1 y_2} u_1 u_2)$$

This is well-defined on $X \times X$ (independent of choice of coordinate chart) [Exercise]

Facts about $\cdot$

1. $\cdot$ is associative [Exercise]
2. $(0, 0, 1)$ is the identity element
3. $(x, y, u)^{-1} = (-x, -y, e^{ixy} u^{-1})$

Thus $(X, \cdot)$ is a group.

Now $\cdot$ and inv are smooth in each coordinate chart (by inspection), so are smooth.

Hence $G = (X, \mathcal{T}_X, |\mathcal{A}_X|, \cdot)$ is a Lie group.

III. EVERY MATRIX LIE GROUP IS A LIE GROUP

Every matrix Lie group G_M is a smooth embedded submanifold of $M_n(\mathbb{C})$ and hence a Lie group.

Idea of Proof:

For each point in G_M, take "small enough" neighborhood on which log is defined to map neighborhood to Lie algebra $\mathfrak{g}$, which is a vector subspace of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$.

To do this formally, we need some facts.

Definition: $||\underset{\sim}{X}||_2 = \left(\sum_{k=1}^{n} \sum_{l=1}^{n} |X_{kl}|^2 \right)^{\frac{1}{2}}$ for $\underset{\sim}{X} \in M_n(\mathbb{C})$

Proposition 1: If $\underset{\sim}{X} \in M_n(\mathbb{C})$ with $||\underset{\sim}{X}|| < \ln 2$, then $\mathit{log}(e^{\underset{\sim}{X}})$ is defined and $\mathit{log}(e^{\underset{\sim}{X}}) = \underset{\sim}{X}$.

[Exercise: See Theorem 2.7 in "Lie Groups, Lie Algebras, and Representations", by Brian Hall]

Definition: Let $\varepsilon \in (0, \ln 2)$. Then let $U_\varepsilon = \{\underset{\sim}{X} \in M_n(\mathbb{C}) : ||\underset{\sim}{X}|| < \varepsilon\}$ and $V_\varepsilon = \mathit{exp}(U_\varepsilon)$

Note: By Proposition 1, V_ε is open in $M_n(\mathbb{C})$

Proposition 2: Suppose $G_M \subseteq GL(n, \mathbb{C})$ is a matrix Lie group with Lie algebra $\mathfrak{g}$. Then there exists $\varepsilon \in (0, \ln 2)$ such that for all $\underset{\sim}{A} \in V_\varepsilon$, we have $\underset{\sim}{A} \in G_M \Leftrightarrow \mathit{log}\, \underset{\sim}{A} \in \mathfrak{g}$.

[Exercise: See Theorem 2.27 in "Lie Groups, Lie Algebras, and Representations", by Brian Hall]

Proposition 3: Every matrix Lie group G_M is a smooth embedded submanifold of $M_n(\mathbb{C})$ and hence a Lie group

Proof:

Let $\varepsilon \in (0, \ln 2)$.

Let $\mathcal{T}_{G_M}$ be the subspace topology on G_M.

Let $\underset{\sim}{A_0} \in G_M$.

Let $\underset{\sim}{C} = \underset{\sim}{0}$.

Then $\underset{\sim}{A_0} = \underset{\sim}{A_0}\,\underset{\sim}{\delta} = \underset{\sim}{A_0}\,\mathscr{exp}(\underset{\sim}{C}) \in \underset{\sim}{A_0}\,\mathscr{exp}(U_\varepsilon) = \underset{\sim}{A_0} V_\varepsilon$.

Since V_ε is open in $M_n(\mathbb{C})$ and multiplication by $\underset{\sim}{A_0}$ is a homeomorphism onto $\underset{\sim}{A_0} V_\varepsilon$, $\underset{\sim}{A_0} V_\varepsilon$ is open in $M_n(\mathbb{C})$.

Thus $\underset{\sim}{A_0} V_\varepsilon$ is an open neighborhood of $\underset{\sim}{A_0}$.

Note: $\underset{\sim}{X} \in \underset{\sim}{A_0} V_\varepsilon \Leftrightarrow \underset{\sim}{A_0}^{-1}\underset{\sim}{X} \in V_\varepsilon$, and by Proposition 2, $\underset{\sim}{A_0}^{-1}\underset{\sim}{X} \in V_\varepsilon \Leftrightarrow \mathscr{log}(\underset{\sim}{A_0}^{-1}\underset{\sim}{X}) \in \mathfrak{g}$

Then define $\varphi_{\underset{\sim}{A_0}} : \underset{\sim}{A_0} V_\varepsilon \to \mathfrak{g}$ by $\varphi_{\underset{\sim}{A_0}}(\underset{\sim}{X}) = \mathscr{log}(\underset{\sim}{A_0}^{-1}\underset{\sim}{X})$.

Then, by Proposition 2, φ is a well-defined homeomorphism. [Exercise]

Now $\mathfrak{g} \subseteq M_n(\mathbb{C})$, and $\mathfrak{g}$ is a vector space, so $\mathfrak{g}$ is a vector subspace of $M_n(\mathbb{C})$.

Let $\{v_1, \ldots v_k\}$ be a basis for $\mathfrak{g}$.

Extend $\{v_1, \ldots v_k\}$ to a basis $\{v_1, \ldots v_{n^2}\}$ for $M_n(\mathbb{C})$.

Let $\eta : M_n(\mathbb{C}) \to \mathbb{R}^{2n^2}$ be defined by

$$\eta\left(\sum_{i=1}^{n^2} a_i v_i\right) = (\Re e\, a_1, \Im m\, a_1, \Re e\, a_2, \Im m\, a_2, \ldots, \Re e\, a_n, \Im m\, a_n)$$

Then η is a linear isomorphism. [Exercise]

Furthermore, $\eta|_{\mathfrak{g}}^{\eta(\mathfrak{g})} : \mathfrak{g} \to \mathbb{R}^{2k} \times \{0\}^{2n^2-2k}$ is a linear isomorphism. [Exercise]

Let $\Phi_{\underset{\sim}{A_0}} = \eta|_{\mathfrak{g}}^{\eta(\mathfrak{g})} \circ \varphi_{\underset{\sim}{A_0}}$.

Then $\Phi_{\underset{\sim}{A_0}}(\underset{\sim}{A_0} V_\varepsilon) = \mathbb{R}^{2k} \times \{0\}^{2n^2-2k} \subseteq \mathbb{R}^{2n^2}$, so is a smooth embedded submanifold of $M_n(\mathbb{C})$.

Then $(\underset{\sim}{A_0} V_\varepsilon, \mathbb{R}^{2k}, \pi_{2k} \circ \Phi_{\underset{\sim}{A_0}})$ is a chart for $\underset{\sim}{A_0}$.

Let $\mathcal{A} = \{(\underset{\sim}{B}V_\varepsilon, \mathbb{R}^{2k}, \pi_{2k} \circ \Phi_{\underset{\sim}{B}}) : \underset{\sim}{B} \in G_M\}$.

Then $(G_M, \mathcal{T}_{G_M}, |\mathcal{A}|, \cdot)$, where $\cdot$ is standard matrix multiplication, is a Lie group.

IV. NOT EVERY LIE GROUP IS A MATRIX LIE GROUP

In fact, we will show even more.

Namely, not every Lie group is **algebraically isomorphic** to a matrix Lie group!

Nilpotent Matrix Lemma

Definition: A matrix $\underset{\sim}{X} \in M_n(\mathbb{R})$ is called **nilpotent** if there exists $k \in \mathbb{N}$ such that $\underset{\sim}{X}^k = \underset{\sim}{0}$.

Lemma: If $\underset{\sim}{X} \in M_n(\mathbb{R})$ is a nonzero nilpotent matrix, then for all nonzero real numbers t, $e^{t\underset{\sim}{X}} \neq \underset{\sim}{\delta}$.

Proof:

Let $\underset{\sim}{X} \neq 0$ be a nilpotent matrix, and suppose, by way of contradiction, there exists $t_0 \in \mathbb{R}$ such that $t_0 \neq 0$ and $e^{t_0\underset{\sim}{X}} = \underset{\sim}{\delta}$.

Since $\underset{\sim}{X}$ is nilpotent, there exists $k \in \mathbb{N}$ such that $\underset{\sim}{X}^k = \underset{\sim}{0}$.

Let $t \in \mathbb{R}$.

Then

$$\begin{aligned} e^{t\underset{\sim}{X}} &= \underset{\sim}{\delta} + t\underset{\sim}{X} + \frac{(t\underset{\sim}{X})^2}{2!} + \cdots + \frac{(t\underset{\sim}{X})^k}{k!} + \frac{(t\underset{\sim}{X})^{k+1}}{(k+1)!} + \ldots \\ &= \underset{\sim}{\delta} + t\underset{\sim}{X} + \frac{t^2\underset{\sim}{X}^2}{2!} + \cdots + \frac{t^{k-1}\underset{\sim}{X}^{k-1}}{(k-1)!} \end{aligned}$$

Let c_{ij}^l be the ijth entry of $\underset{\sim}{X}^l$.

Then $(e^{t\underset{\sim}{X}})_{ij} = \delta_{ij} + c_{ij}^1 t + \frac{c_{ij}^2}{2!}t^2 + \cdots + \frac{c_{ij}^{k-1}}{(k-1)!}t^{k-1}$.

Hence there exists polynomials $p_{ij}(t)$ such that $(e^{t\underset{\sim}{X}})_{ij} = p_{ij}(t)$. (1)

Now let $m \in \mathbb{N}$.

Then $(e^{t_0 \underset{\sim}{X}})^m = \underset{\sim}{\delta}^m$, so $e^{mt_0 \underset{\sim}{X}} = \underset{\sim}{\delta}$.

Then $(e^{mt_0 \underset{\sim}{X}})_{ij} = \rho_{ij}(mt_0)$, so $\rho_{ij}(mt_0) = \delta_{ij}$.

Let $\tilde{\rho_{ij}}(t) = \rho_{ij}(t) - \delta_{ij}$.

Now, if by way of contradiction, $\rho_{ij}(t)$ is nonconstant, then $\tilde{\rho_{ij}}(t)$ is nonconstant and has roots mt_0, for all $m \in \mathbb{N}$.

Thus $\tilde{\rho_{ij}}(t)$ has infinitely many roots, violating the Fundamental "N-th root" Theorem of Algebra.

Thus $\rho_{ij}(t)$ is constant, i.e. there exists $c_{ij} \in \mathbb{R}$ such that $\rho_{ij}(t) = c_{ij}$.

Thus, by (1), for all $t \in \mathbb{R}$, $(e^{t\underset{\sim}{X}})_{ij} = c_{ij}$.

Letting $\underset{\sim}{C} = [c_{ij}]_{i,j=1,1}^{n,n}$, we have $e^{t\underset{\sim}{X}} = \underset{\sim}{C}$.

Then $\frac{d}{dt}(e^{t\underset{\sim}{X}}) = \underset{\sim}{0}$, so $e^{t\underset{\sim}{X}}\underset{\sim}{X} = \underset{\sim}{0}$.

Since this holds for all $t \in \mathbb{R}$, it holds for $t = 0$, so $e^{\underset{\sim}{0}}\underset{\sim}{X} = \underset{\sim}{0}$.

Thus $\underset{\sim}{\delta}\underset{\sim}{X} = \underset{\sim}{0}$, so $\underset{\sim}{X} = \underset{\sim}{0}$, a contradiction.

Relate the Heisenberg matrix Lie group H to the Lie group G

Define $\Phi : H \to G$ by $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mapsto (a, c, e^{ib})$

Then Φ is a surjective homomorphism. [Exercise]

Then $\mathcal{ker}\,\Phi = \left\{ \begin{bmatrix} 1 & 0 & 2\pi n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\} = \{e^{2\pi n \underset{\sim}{B}} : n \in \mathbb{Z}\}$ [Exercise]

Let $N = \mathcal{ker}\,\Phi$.

By the 1st Isomorphism Theorem, $\frac{H}{N} \cong G$.

Definition: Φ is said to be a **Lie group homomorphism** from Lie group $G_L = (X_1, \mathcal{T}_1, |\mathcal{A}_1|, \cdot_1)$ to Lie group $H_L = (X_2, \mathcal{T}_2, |\mathcal{A}_2|, \cdot_2)$ if the following two conditions are met:

1. $\Phi : (X_1, \mathcal{T}_1, |\mathcal{A}_1|) \to (X_2, \mathcal{T}_2, |\mathcal{A}_2|)$ is smooth

2. $\Phi : (X_1, \cdot_1) \to (X_2, \cdot_2)$ is a homomorphism.

Then we write $\Phi : G_L \to H_L$.

Definition: Let G_L be a Lie group. Let $\mathfrak{g}$ be any (unrelated) Lie algebra. Then a **finite-dimensional complex representation** of G_L (resp. $\mathfrak{g}$) is a Lie group homomorphism $\Pi : G \to GL(n, \mathbb{C})$ (resp. Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$).

If Π (resp. π) is injective, then we say that Π (resp. π) is **faithful**.

Theorem: Let Σ be any finite dimensional representation of H. If $N \subseteq \ker \Sigma$, then $Z(H) \subseteq \ker \Sigma$

Proof:

Let Σ be any finite dimensional representation of H.

Let $\sigma : \mathfrak{h} \to \mathfrak{gl}(n, \mathbb{R})$ be defined by $\sigma(\underset{\sim}{X}) = \frac{d}{dt}\left[\Sigma(e^{t\underset{\sim}{X}})\right]\Big|_{t=0}$.

Then σ is a finite dimensional representation of $\mathfrak{h}$, and $\Sigma(e^{\underset{\sim}{X}}) = e^{\sigma(\underset{\sim}{X})}$. [Exercise]

Since σ is Lie algebra homomorphism,

$$[\sigma(\underset{\sim}{A}), \sigma(\underset{\sim}{C})] = \sigma(\underset{\sim}{B}) \text{ and } [\sigma(\underset{\sim}{A}), \sigma(\underset{\sim}{B})] = [\sigma(\underset{\sim}{C}), \sigma(\underset{\sim}{B})] = \underset{\sim}{0}.$$

Let $\underset{\sim}{F} = \sigma(\underset{\sim}{B})$.

Let $\{\lambda_1, \ldots, \lambda_n\}$ be the eigenvalues for $\underset{\sim}{F}$ and F the associated linear operator.

Let $V_{\lambda_i} = \{\underset{\sim}{v} \in \mathbb{C}^n : (\underset{\sim}{F} - \lambda_i \underset{\sim}{\delta})^k \underset{\sim}{v} = \underset{\sim}{0} \text{ for some } k\}$ (generalized eigenspace).

Let $\underset{\sim}{v} \in V_{\lambda_i}$ for some $i \in \{1, \ldots, n\}$.

Then $(\underset{\sim}{F} - \lambda_i \underset{\sim}{\delta})^k \underset{\sim}{F}\underset{\sim}{v} = \underset{\sim}{F}(\underset{\sim}{F} - \lambda_i \underset{\sim}{\delta})^k \underset{\sim}{v} = \underset{\sim}{F} \cdot \underset{\sim}{0} = \underset{\sim}{0}$, so $\underset{\sim}{F}\underset{\sim}{v} \in V_{\lambda_i}$.

Hence V_{λ_i} is invariant under F.

Let $F_{\lambda_i} = F|_{V_{\lambda_i}}$.

Note: $\underset{\sim}{F}_{\lambda_i} - \lambda_i \underset{\sim}{\delta}$ is nilpotent.

Now let λ be an eigenvalue of $\underset{\sim}{F} = \sigma(\underset{\sim}{B})$.

Since $\sigma(\underset{\sim}{A})$ and $\sigma(\underset{\sim}{C})$ commute with $\sigma(\underset{\sim}{B})$, they also leave V_λ invariant. [Exercise]

Now $\mathit{tr}(\sigma(\underset{\sim}{B})|_{V_\lambda}) = \mathit{tr}([\sigma(\underset{\sim}{A})|_{V_\lambda}, \sigma(\underset{\sim}{C})|_{V_\lambda}]) = \underset{\sim}{0}$, since the trace of a commutator is zero.

However, $\mathit{tr}(\sigma(\underset{\sim}{B})|_{V_\lambda}) = \mathit{tr}(\lambda \underset{\sim}{\delta}|_{V_\lambda}) = \lambda\, \mathit{tr}(\underset{\sim}{\delta}|_{V_\lambda}) = \lambda\, \mathit{dim}(V_\lambda)$.

Thus $\lambda\, \mathit{dim}(V_\lambda) = 0$.

Now $\mathit{dim}(V_\lambda) \neq 0$, since λ is an eigenvalue, so $\lambda = 0$.

Hence, for all i, $\underset{\sim}{F}_{\lambda_i} - \lambda_i \underset{\sim}{\delta} = \underset{\sim}{F}_{\lambda_i}$.

Thus $\underset{\sim}{F}_{\lambda_i}$ is nilpotent for each i.

Fact: $\mathbb{C}^n = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}$ [Exercise]

Thus $\underset{\sim}{F} = \sigma(\underset{\sim}{B})$ is nilpotent.

Now $N = \{e^{2\pi n \underset{\sim}{B}} : n \in \mathbb{Z}\} \subseteq \mathit{ker}\, \Sigma$ (hypothesis), so for all $n \in \mathbb{Z}$, $\Sigma(e^{2\pi n \underset{\sim}{B}}) = \underset{\sim}{\delta}$.

Hence, $e^{2\pi n \sigma(\underset{\sim}{B})} = \underset{\sim}{\delta}$ for all $n \in \mathbb{Z}$. (1)

If, by way of contradiction, $\sigma(\underset{\sim}{B}) \neq \underset{\sim}{0}$, then $\sigma(\underset{\sim}{B})$ is a nonzero nilpotent matrix, so by the Nilpotent Matrix Lemma, $e^{t\sigma(\underset{\sim}{B})} \neq \underset{\sim}{\delta}$ for all $t \in \mathbb{R}$ with $t \neq 0$, which contradicts (1).

Thus $\sigma(\underset{\sim}{B}) = \underset{\sim}{0}$.

Now let $X \in Z(H)$.

Then there exists $t \in \mathbb{R}$ such that $X = e^{t\underset{\sim}{B}}$.

Then $\Sigma(\underset{\sim}{X}) = \Sigma(e^{t\underset{\sim}{B}}) = e^{t\sigma(\underset{\sim}{B})} = e^{t \cdot \underset{\sim}{0}} = \underset{\sim}{\delta}$.

Thus $X \in \mathscr{ker}\, \Sigma$, so $Z(H) \subseteq \mathscr{ker}\, \Sigma$.

Proposition: The Lie group G has no faithful finite dimensional representations.

Proof

Suppose $\Psi : G \to GL(n, \mathbb{C})$ is a finite dimensional representation of G.

Then let $\Sigma = \Psi \circ \Phi : H \to GL(n, \mathbb{C})$.

Then Σ is a finite dimensional representation of H.

Let $\underset{\sim}{X} \in N = \mathscr{ker}\, \Phi$. Then $\Phi(\underset{\sim}{X}) = \underset{\sim}{\delta}$.

Then $\Sigma(\underset{\sim}{X}) = \Psi(\Phi(\underset{\sim}{X})) = \Psi(\underset{\sim}{\delta}) = \underset{\sim}{\delta}$, so $\underset{\sim}{X} \in \mathscr{ker}\, \Sigma$.

Thus $N \subseteq \mathscr{ker}\, \Sigma$.

Then, by the above Theorem, $Z(H) \subseteq \mathscr{ker}\, \Sigma$.

Since $Z(H)$ is nontrivial, $\mathscr{ker}\, \Sigma$ is nontrivial, so Σ is not injective.

Thus G has no faithful finite dimensional representation.

Now we show that G is not isomorphic to any matrix Lie group:

Assume, by way of contradiction, that there exists an isomorphism $\eta : G \to G_M$ for some matrix Lie group G_M. Then $\mathscr{inc} : G_M \to GL(n, \mathbb{C})$ is an injective Lie group homomorphism.

Hence $\mathscr{inc} \circ \eta : G \to GL(n, \mathbb{C})$ is an injective Lie group homomorphism, so $\mathscr{inc} \circ \eta$ is a faithful finite dimensional representation, which is a contradiction to the above proposition.

Thus G is not isomorphic to any matrix Lie group.

Note: Since $G \cong \frac{H}{N}$ and since H is a matrix Lie group, we see that matrix Lie groups are not preserved by taking quotients.