Supplement 2 for Section 2.6

This material should come Oblique after Asymptotes on page 89.

The long division algorithm used to find oblique asymptotes can also be used to find other curves that are approached asymptotically by the function at $+\infty$ or $-\infty$. Specifically, suppose for the rational function $P(x) = \dfrac{N(x)}{D(x)}$, the degree of N is larger than that of D. Then the division algorithm produces a quotient polynomial Q and a remainder polynomial R whose degree is less than that of D such that $P(x) = Q(x) + \dfrac{R(X)}{D(x)}$. As was noted earlier in the chapter, for a quotient of polynomials whose denominator has degree strictly smaller than that of the numerator, the quotient has limit 0 as $x \to \pm\infty$. Consequently $\lim_{x\to\pm\infty} P(x) = Q(x)$. That is, the function P is approximately equal to the polynomial Q for large values of $|x|$. For example let $P(x) = \dfrac{x^5 + 2x^4 - x^3 - x - x + 1}{x^3 - 1}$. Then

$$
\begin{array}{r|l}
 & \quad x^2 + 2x - 1 \\
\hline
x^3 - 1 & x^5 + 2x^4 - x^3 \qquad - 2x + 1 \\
 & -x^5 \qquad\quad + x^2 \\
 & \quad 2x^4 - x^3 + x^2 - 2x \\
 & \quad -2x^4 \qquad + 2x \\
 & \qquad -x^3 + x^2 \qquad + 1 \\
 & \qquad x^3 \qquad\quad - 1 \\
 & \qquad\quad x^2
\end{array}
$$

Thus $P(x) = x^2 + 2x - 1 + \frac{x^2}{x^3-1}$ and hence $P(x) \approx x^2 + 2x - 1$ for large values of $|x|$.
Generally speaking it is very useful to be able to approximate a function by a polynomial because evaluating a polynomial is a purely arithmetic process. Later we well learn how to approximate certain functions near a number a in their domains by a polynomial.