Supplement 5 for Section 3.6

This material should come at the end of page 146.

The Power Rule will now be extended to rational exponents using the Chain Rule. First recall that a rational number is one that can be written as a quotient of two integers. For example $\frac{8}{-6}$. The representation isn't unique. In this case $\frac{8}{-6} = \frac{-4}{3}$. The representation $r = \frac{m}{n}$ is unique if n is required to be positive and if m and n have no common divisors except for 1. In that case if n is even, the domain of the function $f(x) = x^{\frac{m}{n}}$ is $[0, \infty)$ if $m \geq 0$ or $(0, \infty)$ otherwise. But if n is odd, then the domain of f is $(-\infty, \infty)$ if $m \geq 0$ or $(-\infty, 0) \cup (0, \infty)$ otherwise.

More General Power Rule. *Let r be a rational number. Then for any x in the domain of x^r*

$$\frac{d}{dx} x^r = r x^{r-1}$$

Proof. Choose a positive integer, n, and an integer, m, so that $r = \frac{m}{n}$. Then

$$\begin{aligned}
\frac{d}{dx} x^r &= \frac{d}{dx} x^{\frac{m}{n}} = \frac{d}{dx} \left(x^{\frac{1}{n}}\right)^m \\
&= m\left(x^{\frac{1}{n}}\right)^{m-1} \frac{d}{dx} x^{\frac{1}{n}} \quad \text{(by the Chain Rule)} \\
&= m\left(x^{\frac{m-1}{n}}\right) \frac{1}{n} x^{\frac{1}{n}-1} \\
&= \frac{m}{n} x^{\frac{m}{n} - \frac{1}{n} + \frac{1}{n} - 1} = r x^{r-1}
\end{aligned}$$

□