Supplement 8 for Section 5.1

The material here replaces the material on page 259 starting with, "Riemann Sums".

The material just developed is used to show that the area under the graph of the function $f(x) = x^2$ on the interval $[0, b]$ for any $b > 0$ is $\frac{1}{3}b^3$. First let n be any positive integer and divide the interval $[0, b]$ into n subintervals of equal length; namely, $\frac{b}{n}$. The intervals will be $[(k-1)\frac{b}{n}, k\frac{b}{n}] = I_{n,k}$ for $k = 1, 2, 3, \ldots n$. The task is to show that for any choice of the numbers $c_k \in I_{n,k}$, $\lim_{n\to\infty} \frac{b}{n}\sum_{k=1}^{n} c_k^2 = \frac{b^3}{3}$. Suppose $c_k \in I_{n,k}$. Then For each $k = 1, 2, 3, \ldots, n$ we have $(k-1)\frac{b}{n} \le c_k \le k\frac{b}{n}$. Consequently it suffices to consider just two choices; namely, $c_k = (k-1)\frac{b}{n}$ and $c_k = k\frac{b}{n}$. The sum for any other choices for the numbers c_k will lie between the sum for the first choice and the sum for the second choice. We begin with the first choice.

$$\begin{aligned}
\frac{b}{n}\sum_{k=1}^{n}\left((k-1)\frac{b}{n}\right)^2 &= \left(\frac{b}{n}\right)^3 \sum_{k=1}^{n}(k-1)^2 = \left(\frac{b}{n}\right)^3 \sum_{k=2}^{n}(k-1)^2 \\
&= \left(\frac{b}{n}\right)^3 \sum_{k=1}^{n-1} nk^2 \text{ (replacing } k-1 \text{ with } k) \\
&= \left(\frac{b}{n}\right)^3 \frac{(n-1)n(2(n-1)+1)}{6}
\end{aligned}$$

(See the formulas near the bottom of page 258.)

$$= b^3 \frac{(1-\frac{1}{n})(2-\frac{1}{n})}{6}$$

Consequently $\lim_{n\to\infty} \frac{b}{n}\sum_{k=1}^{n}\left((k-1)\frac{b}{n}\right)^2 = \lim_{n\to\infty} b^3\frac{(1-\frac{1}{n})(2-\frac{1}{n})}{6} = \frac{1}{3}b^3$.

In a similar fashion it can be shown that $\lim_{n\to\infty} \frac{b}{n}\sum_{k=1}^{n}\left(k\frac{b}{n}\right)^2 = \frac{1}{3}b^3$. Because the limit for the two special choices of the numbers c_k are identical, the limit will be the common value for any choice for the numbers c_k. Thus the area under the graph of $f(x) = x^2$ on $[0, b]$ is $\frac{b^3}{3}$.

Using a very similar procedure, it can be shown that the area under the graph of $f(x) = x$ on the interval $[0, b]$ for any $b > 0$ is $\frac{1}{2}b^2$, which can also be determined by geometric considerations.