Supplement 9 for Section 5.3

The material here replaces Section 5.3, pages 262–270.

In the previous two sections we saw how to find the area under the graph of a function and computed that area in one nontrivial case. In this section the same procedure is applied to any function; not just those whose graphs lie above the x axis. The result is called the *definite integral*,

Definition 1. Let f be a function defined on a closed interval $[a, b]$. Then f is (Riemann) integrable on $[a, b]$ means there is a number, denoted by $\int_a^b f(x)\,dx$ (called the definite or Riemann integral of f), such that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(c_k)$$

for any selection of the numbers $c_k \in [a + (k-1)\frac{b-a}{n}, a + k\frac{b-a}{n}]$ for each n.

For example it has been shown that the function $f(x) = x^2$ is integrable from 0 to b for any $b > 0$ and $\int_0^b x^2\,dx = \frac{1}{3}b^3$. Also $\int_0^b x\,dx = \frac{1}{2}b^2$ for any $b > 0$. It is easy to show that for any number C, the function $f(x) = C$ is integrable from a to b and that $\int_a^b C\,dx = C(b-a)$.

It could be easily concluded from the complicated nature of this definition, that only very special functions are integrable, but the following theorem asserts that there is a very large collection of integrable functions.

Theorem 1. *Let f be continuous on $[a, b]$. Then f is integrable from a to b.*

The proof of this assertion required a deeper understanding of the nature of the set of real numbers than we have at this point, but it does assure us that there is a substantial collection of integrable functions. The converse of the theorem is false; that is, there are integrable functions that aren't continuous everywhere on $[a, b]$. For example, changing the value of an integrable function at one point results in a different integrable function with the same integral. However, changing an integrable function at too many points may result in a function that isn't integrable. For example the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational} \end{cases}$$

isn't integrable from say 0 to 1 even though it differs from the constant (and hence integrable) function $g(x) = 1$ at just the rational numbers.

The symbol used to denote the definite integral $\displaystyle\int$ is an elongated version of the letter, "S", because the definite integral can be thought of as a sums of infinitely many terns. The symbol dx indicates which symbol in the definition of f is the variable. For example, if $f(x)$ were defined as sx^3, we would know that we were to regard s as a number and not a variable.

Properties of the Definite Integral

Some general properties of the definite integral are given in the next theorem.

Theorem 2. *Let f and g be integrable on $[a, b]$ and let C be a number.*

1. *Then $f + g$ is integrable on $[a, b]$ and $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$*

2. *Then* Cf *is integrable on* $[a,b]$ *and* $\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$

3. *Then* $f \cdot g$ *is integrable on* $[a,b]$*, but there is no formula for* $\int_a^b f(x)g(x)\,dx$

4. *Then* f *is bounded; that is, there are two numbers* $m < M$ *such that for each* $x \in [a,b]$, $m \le f(x) \le M$

5. *If* $f(x) \le g(x)$ *for all* $x \in [a,b]$*, then* $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

6. *If* $f(x) < g(x)$ *for all* $x \in [a,b]$*,, then* $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$

7. *Then* $|f|$ *is integrable from* a *to* b *and* $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

From 1. and 2. it's easy to prove, with the assumptions of the theorem, that $f - g$ is integrable on $[a,b]$ and $\int_a^b (f-g)(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$. Assertions 1., 2., and 5. are fairly easy to prove but 4. and 6. are quite a bit more difficult. Numbers 3. and 7. require the same depth of knowledge as is needed to prove Theorem 1.

Definition 2. Let f be defined on the interval $[a,b]$ with $f(x) \ge 0$ for all $x \in [a,b]$. The region $R = \{(x,y); x \in [a,b] \text{ and } 0 \le y \le f(x)\}$ is the region under the graph of f from a to b. The area of R is $\int_a^b f(x)\,dx$.

The next property of integrals is motivated of the geometric interpretation in terms of area.

Theorem 3. *Let* $a \le b \le c$ *be numbers and let* f *be a function. Then* f *is integrable from* a *to* c *if and only if* f *is integrable from* b *to* b *and from* b *to* c*. Moreover* $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

Actually with the definition of $\int_b^a f(x)\,dx$ as $-\int_a^b f9x)\,dx$, the formula holds regardless of the relationship between a, b and c.

Average Value

By a minor change in the formulation of the definition of the definite integral, the *average value* of a function on an interval can be defined. Note that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(c_k)$$

and then observe that $\frac{1}{n}\sum_{k=1}^{n} f(c_k)$ is the average of n values of the function at more or less evenly distributed numbers in the interval. By taking the limit as $n \to \infty$ the number of terms gets large and could be interpreted as approaching the average value of the function on that interval.

Definition 3. Let f be integrable on the interval $[a,b]$. The the average value of f on that interval is defined to be $\frac{1}{b-a}\int_a^b f(x)\,dx$.

If the integrand f is continuous, then the average value is a value of the function as is asserted in the following theorem.

Theorem 4. *Let* f *be continuous on* $[a,b]$*. Then there is a number* $c \in [a,b]$ *such that* $\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$.

Proof. Because f is continuous on $[a,b]$, by the Extreme Value Theorem there are $s, t \in [a,b]$ such that for each $x \in [a,b]$ $f(s) \le f(x) \le f(t)$. Consequently by 5. of Theorem 2

$$f(s) = \frac{1}{b-a}\int_a^b f(s)\,dx \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{b-a}\int_a^b f(t)\,dx = f(t)$$

Now by the Intermediate Value Theorem there is a $c \in [a,b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$. $\square$