Chapter 8
Real Numbers

Mathematical study and research are very suggestive of mountaineering. Whymper made several efforts before he climbed the Matterhorn in the 1860's and even then it cost the life of four of his party. Now, however, any tourist can be hauled up for a small cost, and perhaps does not appreciate the difficulty of the original ascent. So in mathematics, it may be found hard to realise the great initial difficulty of making a little step which now seems so natural and obvious, and it may not be surprising if such a step has been found and lost again.
Louis Joel Mordell (1888–1972)

Before You Get Started. Just like the integers, the real numbers, which ought to include the integers but also numbers like $\frac{1}{3}$, $-\sqrt{2}$, and π, will be defined by a set of axioms. From what you know about real numbers, what should this set of axioms include? How should the axioms differ from those of Chapters 1 and 2?

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We start all over again. You have used the real numbers in calculus. You have pictured them as points on an x-axis or a y-axis. You have probably been told that there is a bijection between the set of points on the x-axis and the set of all real numbers. Even if this was not made explicit in your calculus course, it was implied when you gave a real-number label to an arbitrary point on the x-axis, or when you assumed that there is a point on the x-axis for every real number.

The exact relationship between the integers and the real numbers will require careful discussion in Chapter 9.

Intuitively, you are familiar with many real numbers: examples are $-\sqrt{2}$, π, and $6e$. You probably thought of the integers as examples of real numbers: you calibrated the x-axis by marking two points as “0” and “1”, thus defining one unit of length; and, with that calibration, you knew which point on the x-axis should get the label “7” and which should get the label “-4”.

We are now going to rebuild your knowledge of the real numbers. In the first stage, which is this chapter, we will define the real numbers by means of axioms, just as we did with the integers in Part I. And as we did with the set of integers $\mathbf{Z}$, we will assume without proof that a set $\mathbf{R}$ satisfying our axioms exists.

8.1 Axioms

We assume that there exists a set, denoted by $\mathbf{R}$, whose members are called **real numbers**. This set $\mathbf{R}$ is equipped with binary operations $+$ and $\cdot$ satisfying Axioms 8.1–8.5, 8.26, and 8.52 below.

Axiom 8.1. *For all $x, y, z \in \mathbf{R}$:*

(i) $x+y=y+x$.

(ii) $(x+y)+z=x+(y+z)$.

(iii) $x\cdot(y+z)=x\cdot y+x\cdot z$.

(iv) $x\cdot y=y\cdot x$.

(v) $(x\cdot y)\cdot z=x\cdot(y\cdot z)$.

The product $x\cdot y$ is often written xy.

Axiom 8.2. *There exists a real number 0 such that for all $x\in\mathbf{R}$, $x+0=x$.*

Axiom 8.3. *There exists a real number 1 such that $1\neq 0$ and whenever $x\in\mathbf{R}$, $x\cdot 1=x$.*

Axiom 8.4. *For each $x\in\mathbf{R}$, there exists a real number, denoted by $-x$, such that $x+(-x)=0$.*

Axiom 8.5. *For each* $x \in \mathbf{R} - \{0\}$*, there exists a real number, denoted by* x^{-1}*, such that* $x \cdot x^{-1} = 1$.

Proposition 8.6. *For all* $x, y \in \mathbf{R} - \{0\}$*,* $(xy)^{-1} = x^{-1} y^{-1}$.

Proposition 8.7. *Let* $x, y, z \in \mathbf{R}$ *and* $x \neq 0$*. If* $xy = xz$ *then* $y = z$.

Proof. Assume $x, y, z \in \mathbf{R}$, $x \neq 0$, and $xy = xz$. By Axiom 8.5, there exists x^{-1}, and thus

$$\begin{aligned} x^{-1}(xy) &= x^{-1}(xz) \\ \left(x^{-1}x\right) y &= \left(x^{-1}x\right) z \\ \left(x x^{-1}\right) y &= \left(x x^{-1}\right) z \\ 1 \cdot y &= 1 \cdot z \\ y &= z\,. \end{aligned}$$

Here we have used Axioms 8.1(v), 8.1(iv), 8.5, and 8.3. □

Proposition 8.7 is the **R**-analogue of Axiom 1.5 for **Z**: the proposition asserts that the cancellation property described in Axiom 1.5 also holds in **R**. And since Axioms 8.1–8.4 are the same as Axioms 1.1–1.4, any proposition we proved about **Z** using only Axioms 1.1–1.5 is also true for **R**, with an identical proof. We will need to refer to some of the real versions of the propositions proved for **Z**; so we state the corresponding propositions for **R** (which again will have the same proof as those for **Z**) in small font.

Proposition 8.8. *If* $m, n, p \in \mathbf{R}$ *then* $(m+n)p = mp + np$.

Proposition 8.9. *If* $m \in \mathbf{R}$*, then* $0 + m = m$ *and* $1 \cdot m = m$.

Proposition 8.10. *Let* $m, n, p \in \mathbf{R}$*. If* $m + n = m + p$*, then* $n = p$.

Proposition 8.11. *Let* $m, x_1, x_2 \in \mathbf{R}$*. If* m, x_1, x_2 *satisfy the equations* $m + x_1 = 0$ *and* $m + x_2 = 0$*, then* $x_1 = x_2$.

Proposition 8.12. *If* $m, n, p, q \in \mathbf{R}$ *then*

(i) $(m+n)(p+q) = (mp+np) + (mq+nq)$.
(ii) $m + (n + (p+q)) = (m+n) + (p+q) = ((m+n)+p) + q$.
(iii) $m + (n+p) = (p+m) + n$.
(iv) $m(np) = p(mn)$.
(v) $m(n + (p+q)) = (mn + mp) + mq$.
(vi) $(m(n+p))\,q = (mn)q + m(pq)$.

Proposition 8.13. *Let* $x \in \mathbf{R}$*. If* x *has the property that for each* $m \in \mathbf{R}$*,* $m + x = m$*, then* $x = 0$.

Proposition 8.14. *Let $x \in \mathbf{R}$. If x has the property that there exists $m \in \mathbf{R}$ such that $m + x = m$, then $x = 0$.*

Proposition 8.15. *For all $m \in \mathbf{R}$, $m \cdot 0 = 0 = 0 \cdot m$.*

Proposition 8.16. *Let $x \in \mathbf{R}$. If x has the property that for all $m \in \mathbf{R}$, $mx = m$, then $x = 1$.*

Proposition 8.17. *Let $x \in \mathbf{R}$. If x has the property that for some nonzero $m \in \mathbf{R}$, $mx = m$, then $x = 1$.*

Proposition 8.18. *For all $m, n \in \mathbf{R}$, $(-m)(-n) = mn$.*

Proposition 8.19.

(i) *For all $m \in \mathbf{R}$, $-(-m) = m$.*
(ii) $-0 = 0$.

Proposition 8.20. *Given $m, n \in \mathbf{R}$ there exists one and only one $x \in \mathbf{R}$ such that $m + x = n$.*

Proposition 8.21. *Let $x \in \mathbf{R}$. If $x \cdot x = x$ then $x = 0$ or 1.*

Proposition 8.22. *For all $m, n \in \mathbf{R}$:*

(i) $-(m+n) = (-m) + (-n)$.
(ii) $-m = (-1)m$.
(iii) $(-m)n = m(-n) = -(mn)$.

Proposition 8.23. *Let $m, n \in \mathbf{R}$. If $mn = 0$, then $m = 0$ or $n = 0$.*

As with $\mathbf{Z}$, we define **subtraction** in $\mathbf{R}$ by

$$x - y := x + (-y) .$$

Proposition 8.24. *For all $m, n, p, q \in \mathbf{R}$:*

(i) $(m-n) + (p-q) = (m+p) - (n+q)$.
(ii) $(m-n) - (p-q) = (m+q) - (n+p)$.
(iii) $(m-n)(p-q) = (mp+nq) - (mq+np)$.
(iv) $m - n = p - q$ *if and only if* $m + q = n + p$.
(v) $(m-n)p = mp - np$.

Alternative notations for $\frac{y}{x}$ are y/x and $y \div x$. Do not confuse the division symbol $/$ with the symbol $|$ which describes the divisibility property of integers introduced in Section 1.2.

Here is a definition that we could not make in $\mathbf{Z}$: We define a new operation on $\mathbf{R}$ called **division** by

$$\frac{y}{x} := y \cdot x^{-1} .$$

Axiom 8.5 does not assert the existence of 0^{-1}; so division is not defined when $x = 0$. In the language of Section 5.4, the division function is

$$\text{division} : \mathbf{R} \times (\mathbf{R} - \{0\}) \to \mathbf{R}, \qquad \text{division}(y, x) = y \cdot x^{-1} .$$

Note that $\frac{1}{x} = 1 \cdot x^{-1} = x^{-1}$, and so we usually write x^{-1} as $\frac{1}{x}$.

Project 8.25. Think about why division by 0 ought not to be defined. Come up with an argument that will convince a friend.

8.2 Positive Real Numbers and Ordering

Axiom 8.26. *There exists a subset* $\mathbf{R}_{>0} \subseteq \mathbf{R}$ *satisfying:*

(i) *If* $x, y \in \mathbf{R}_{>0}$ *then* $x + y \in \mathbf{R}_{>0}$.

(ii) *If* $x, y \in \mathbf{R}_{>0}$ *then* $xy \in \mathbf{R}_{>0}$.

(iii) $0 \notin \mathbf{R}_{>0}$.

(iv) *For every* $x \in \mathbf{R}$, *we have* $x \in \mathbf{R}_{>0}$ *or* $x = 0$ *or* $-x \in \mathbf{R}_{>0}$.

The members of $\mathbf{R}_{>0}$ are called **positive real numbers**. A **negative real number** is a real number that is neither positive nor zero.

Proposition 8.27. *For* $x \in \mathbf{R}$, *one and only one of the following is true:* $x \in \mathbf{R}_{>0}$, $-x \in \mathbf{R}_{>0}$, $x = 0$.

Proposition 8.28. $1 \in \mathbf{R}_{>0}$.

By analogy with the definition of "less than" in $\mathbf{Z}$, we write $x < y$ (x **is less than** y) or $y > x$ (y **is greater than** x) if $y - x \in \mathbf{R}_{>0}$, and we write $x \leq y$ (x **is less than or equal to** y) or $y \geq x$ (y **is greater than or equal to** x) if we also allow $x = y$. The analogy between the $<$ relation on $\mathbf{R}$ and $<$ as previously defined on $\mathbf{Z}$ continues:

Proposition 8.29. *Let* $x, y, z \in \mathbf{R}$. *If* $x < y$ *and* $y < z$ *then* $x < z$.

Proposition 8.30. *For each* $x \in \mathbf{R}$ *there exists* $y \in \mathbf{R}$ *such that* $y > x$.

Proposition 8.31. *Let* $x, y \in \mathbf{R}$. *If* $x \leq y \leq x$ *then* $x = y$.

Proposition 8.32. *For all* $x, y, z, w \in \mathbf{R}$:

(i) *If* $x < y$ *then* $x + z < y + z$.

(ii) *If* $x < y$ *and* $z < w$ *then* $x + z < y + w$.

(iii) *If* $0 < x < y$ *and* $0 < z \leq w$ *then* $xz < yw$.

(iv) *If* $x < y$ *and* $z < 0$ *then* $yz < xz$.

Proposition 8.33. *For each* $x, y \in \mathbf{R}$, *exactly one of the following is true:* $x < y$, $x = y$, $x > y$.

Proposition 8.34. *Let* $x \in \mathbf{R}$. *If* $x \neq 0$ *then* $x^2 \in \mathbf{R}_{>0}$.

Proposition 8.35. *The equation* $x^2 = -1$ *has no solution in* $\mathbf{R}$.

Proposition 8.36. *Let* $x, z \in \mathbf{R}_{>0}$, $y \in \mathbf{R}$. *If* $xy = z$, *then* $y \in \mathbf{R}_{>0}$.

Proposition 8.37. *For all* $x, y, z \in \mathbf{R}$:

(i) $-x < -y$ *if and only if* $x > y$.

(ii) *If* $x > 0$ *and* $xy < xz$ *then* $y < z$.

(iii) *If* $x < 0$ *and* $xy < xz$ *then* $z < y$.

(iv) *If* $x \le y$ *and* $0 \le z$ *then* $xz \le yz$.

Proposition 8.38. $\mathbf{R}_{>0} = \{x \in \mathbf{R} : x > 0\}$.

Proposition 8.39. *If* $x \in \mathbf{R}_{>0}$ *then* $x+1 \in \mathbf{R}_{>0}$.

Proposition 8.40.

(i) $x \in \mathbf{R}_{>0}$ *if and only if* $\frac{1}{x} \in \mathbf{R}_{>0}$.

(ii) *Let* $x, y \in \mathbf{R}_{>0}$. *If* $x < y$ *then* $0 < \frac{1}{y} < \frac{1}{x}$.

After proving this proposition, draw graphs of $y = x^2$ *and* $y = x^3$.

Proposition 8.41. *Let* $x \in \mathbf{R}$. *Then* $x^2 < x^3$ *if and only if* $x > 1$.

8.3 Similarities and Differences

A mathematical system that satisfies Axioms 1.1–1.4 is called a commutative ring.

If you compare Axioms 1.1–1.4 (for $\mathbf{Z}$) with Axioms 8.1–8.4 (for $\mathbf{R}$) you will see that they are identical. They are concerned with addition, subtraction, 0, and 1. It follows that any proposition for $\mathbf{Z}$ that depends only on Axioms 1.1–1.4 is automatically also true for $\mathbf{R}$. In fact, the same holds for $\mathbf{Z}_n$, by Proposition 6.26.

In the same way, Axiom 2.1 and Axiom 8.26 are identical: they concern the positive numbers and ordering. Thus once again we can get “free” theorems for real numbers based on proofs originally given for integers.

Now compare Axiom 1.5 (cancellation) with Axiom 8.5 (multiplicative inverse). As we showed in Proposition 8.7, Axiom 8.5 implies Axiom 1.5. The converse implication is false: for example, the integer 2 does not have a multiplicative inverse in $\mathbf{Z}$.

Another notable difference between $\mathbf{Z}$ and $\mathbf{R}$ involves the existence of a smallest positive element. By Proposition 2.20, the integer 1 is the smallest positive integer. There is no comparable statement for $\mathbf{R}$:

Here we use the same definition for “smallest element” that we used in Section 2.4.

Theorem 8.42. $\mathbf{R}_{>0}$ *does not have a smallest element.*

Proof. Define the real number $2 := 1+1$; by Proposition 8.28, $2 \in \mathbf{R}_{>0}$. Proposition 8.40 implies that $2^{-1} = \frac{1}{2}$ is also positive.

We claim further that $\frac{1}{2} < 1$; otherwise, Proposition 8.32(ii) (with $0 < 1 < 2$ and $0 < 1 \le \frac{1}{2}$) would imply that $1 < 1$, a contradiction.

Thus we have established $0 < \frac{1}{2} < 1$ and can start the actual proof of Theorem 8.42. We will prove it by contradiction. Assume that there exists a smallest element

$s \in \mathbf{R}_{>0}$. Then we can use Proposition 8.32(ii) (with $0 < \frac{1}{2} < 1$ and $0 < s \leq s$) to deduce
$$\frac{1}{2} \cdot s < s .$$
However, $\frac{1}{2} \cdot s \in \mathbf{R}_{>0}$ (by Axiom 8.26(ii)), which contradicts the fact that s is the smallest element in $\mathbf{R}_{>0}$. □

We labeled Theorem 8.42 as a theorem rather than a proposition to emphasize its importance. In many of your advanced mathematics courses—courses with words like *analysis* and *topology* in their titles—the instructor will use Theorem 8.42 regularly. It may not be mentioned explicitly, but it will be used in ε–δ arguments. We will discuss this in more detail in Chapter 10.

Theorem 8.43. *Let $x, y \in \mathbf{R}$ such that $x < y$. There exists $z \in \mathbf{R}$ such that $x < z < y$.*

This theorem implies that the real numbers are "all over the place" in the sense that no matter how close two real numbers are, there are infinitely many real numbers between these two. (See Section 13.1 for the meaning of "infinitely many.")

The analogous statement for $\mathbf{Z}$ is false—this is the content of Corollary 2.22.

The remaining axiom for $\mathbf{Z}$, Axiom 2.15, is concerned with induction; it has no analogue for the real numbers:

Project 8.44. Construct a subset $A \subseteq \mathbf{R}$ that satisfies

(i) $1 \in A$ and

(ii) if $n \in A$ then $n + 1 \in A$,

yet for which $\mathbf{R}_{>0}$ is not a subset of A.

In the next section, we will introduce one more axiom for $\mathbf{R}$, called the Completeness Axiom; it has no useful analogue for $\mathbf{Z}$.

8.4 Upper Bounds

To state our last axiom for $\mathbf{R}$, we need some definitions. Let A be a nonempty subset of $\mathbf{R}$.

(i) The set A is **bounded above** if there exists $b \in \mathbf{R}$ such that for all $a \in A$, $a \leq b$. Any such number b is called an **upper bound** for A.

(ii) The set A is **bounded below** if there exists $b \in \mathbf{R}$ such that for all $a \in A$, $b \leq a$. Any such number b is called a **lower bound** for A.

(iii) The set A is **bounded** if it is both bounded above and bounded below.

(iv) A **least upper bound** for A is a an upper bound that is less than or equal to every upper bound for A.

Least upper bounds are unique if they exist:

Proposition 8.45. *If x_1 and x_2 are least upper bounds for A, then $x_1 = x_2$.*

$\sup(A)$ *is often written as* $\sup A$*, as in Example 8.46. An alternative notation for* $\sup(A)$ *is* $\operatorname{lub}(A)$.

The least upper bound of A is denoted by $\sup(A)$, an abbreviation for **supremum**.

Example 8.46. $\sup\{x \in \mathbf{R} : x < 0\} = 0$.

The least upper bound of a set might not exist. For example:

Proposition 8.47. $\mathbf{R}_{>0}$ *has no upper bound.*

Example 8.48. Consider the sets

$$\{x \in \mathbf{R} : 0 \le x \le 1\}$$

and

$$\{x \in \mathbf{R} : 0 \le x < 1\}.$$

In both cases, the least upper bound is 1. In the first set, the least upper bound lies in the set, while in the second set the least upper bound lies outside. The important fact, illustrated by this example, is that $\sup(A)$ sometimes lies in A but not always. We will say more in the next proposition.

Propositions 8.45 and 8.49 imply that $\max(A)$ *is unique if it exists.*

A real number $b \in A$ is the **maximum** or **largest** element of A if for all $a \in A$, $a \le b$. In this case we write $b = \max(A)$.

Proposition 8.49. *Let $A \subseteq \mathbf{R}$ be nonempty. If* $\sup(A) \in A$ *then* $\sup(A)$ *is the largest element of A, i.e.,* $\sup(A) = \max(A)$. *Conversely, if A has a largest element then* $\max(A) = \sup(A)$ *and* $\sup(A) \in A$.

Proposition 8.50. *If the sets A and B are bounded above and $A \subseteq B$, then* $\sup(A) \le \sup(B)$.

At this point it is useful to define **intervals**. They come in nine types: Let $x < y$. Then

$[x,y]$ *is an example of a* closed *interval;* (x,y) *is an* open *interval; and* $(x,y]$ *is* half open.
Do not confuse the open interval notation with the coordinate description of a point in the plane.

$$\begin{aligned}
[x,y] &:= \{z \in \mathbf{R} : x \le z \le y\}\\
(x,y] &:= \{z \in \mathbf{R} : x < z \le y\}\\
[x,y) &:= \{z \in \mathbf{R} : x \le z < y\}\\
(x,y) &:= \{z \in \mathbf{R} : x < z < y\}\\
(-\infty,x] &:= \{z \in \mathbf{R} : z \le x\}\\
(-\infty,x) &:= \{z \in \mathbf{R} : z < x\}\\
[x,\infty) &:= \{z \in \mathbf{R} : x \le z\}\\
(x,\infty) &:= \{z \in \mathbf{R} : x < z\}\\
(-\infty,\infty) &:= \mathbf{R}.
\end{aligned}$$

Project 8.51. For a nonempty set $B \subseteq \mathbf{R}$ one can define the **greatest lower bound** $\inf(B)$ (for **infimum**) of B. Give the precise definition for $\inf(B)$ and prove that it is unique if it exists. Also define $\min(B)$ and prove the analogue of Proposition 8.49 for greatest lower bounds and minima.

An alternative notation for $\inf(B)$ *is* $\mathrm{glb}(B)$.

Here is the final axiom for the real numbers.

Axiom 8.52 (Completeness Axiom). *Every nonempty subset of* **R** *that is bounded above has a least upper bound.*

This axiom, which concludes our definition of **R**, is stated here only because those referring back later might forget to include it in the list. It needs discussion, indeed a chapter of its own—Chapter 10.

Proposition 8.53. *Every nonempty subset of* **R** *that is bounded below has a greatest lower bound.*

Review Questions. Have you looked carefully at how the axioms for the set of real numbers differ from the axioms for the set of integers? Do you understand the difference between the maximum element of a set of real numbers and the least upper bound of that set?

Weekly reminder: Reading mathematics is not like reading novels or history. You need to think slowly about every sentence. Usually, you will need to reread the same material later, often more than one rereading.

This is a short book. Its core material occupies about 140 pages. Yet it takes a semester for most students to master this material. In summary: read line by line, not page by page.

Chapter 9
Embedding Z in R

I believe that numbers and functions of analysis are not the arbitrary result of our minds; I think that they exist outside of us, with the same character of necessity as the things of objective reality, and we meet them or discover them, and study them, as do the physicists, the chemists and the zoologists.
Charles Hermite (1822–1901), quoted in Morris Kline's *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972, p. 1035.

We have now defined two number systems, $\mathbf{Z}$ and $\mathbf{R}$. Intuitively, we think of the integers as a subset of the real numbers; however, nothing in our axioms tells us explicitly that $\mathbf{Z}$ can be viewed as a subset of $\mathbf{R}$. In fact, at the moment we have no axiomatic reason to think that the integers we named 0 and 1 are the same as the real numbers we named 0 and 1.

Just for now, we will be more careful and write $0_{\mathbf{Z}}$ and $1_{\mathbf{Z}}$ for these special members of $\mathbf{Z}$, and $0_{\mathbf{R}}$ and $1_{\mathbf{R}}$ for the corresponding special members of $\mathbf{R}$. Informally we are accustomed to identifying $0_{\mathbf{Z}}$ with $0_{\mathbf{R}}$ and identifying $1_{\mathbf{Z}}$ with $1_{\mathbf{R}}$. We will justify this here by giving an *embedding* of $\mathbf{Z}$ into $\mathbf{R}$, that is, a function that maps each integer to the corresponding number in $\mathbf{R}$.

Before You Get Started. How could such an embedding function of $\mathbf{Z}$ into $\mathbf{R}$ be constructed? From what you know about functions, what properties will such a function have?

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