Theorem (The Euclidean Algorithm). Let x and y be integers. Then there exist integers q_1, q_2, ..., q_k and a descending sequence of positive integers, r_1, ..., r_k, $r_{k+1} = 0$, such that:

$$x = q_1 y + r_1$$
$$y = q_2 r_1 + r_2$$
$$r_1 = q_3 r_2 + r_3$$
$$\vdots$$
$$r_{k-1} = q_k r_k + 0$$

Furthermore, $\gcd(x, y) = r_k$.

EX. Find a greatest common divisor of 4095 and 165 by using the Euclidean Algorithm.

$$\begin{aligned} 4095 &= 24 \times 165 + 135 && (1) \\ 165 &= 1 \times 135 + 30 && (2) \\ 135 &= 4 \times 30 + 15 && (3) \\ 30 &= 2 \times 15. && (4) \end{aligned}$$

Theorem. Let g be the greatest common divisor of b and c. Then, there exist integers x and y such that $g = bx + cy$.

From the previous example, let's find integers x and y. From the Euclidean Algortihm, by working backward we have

$$\begin{aligned} 15 &= 135 - 4 \times 30 \quad \text{From Eq (3)} \\ &= 135 - 4 \times (165 - 1 \times 135) \quad \text{From Eq (2)} \\ &= 135 - 4 \times 165 + 4 \times 135 \\ &= 5 \times 135 - 4 \times 165 \\ &= 5 \times (4095 - 24 \times 165) - 4 \times 165 \quad \text{From Eq (1)} \\ &= 5 \times 4095 - 5 \times 24 \times 165 - 4 \times 165 \\ &= 5 \times 4095 - 124 \times 165 \end{aligned} \qquad (5)$$

Therefore $x = 5$ and $y = -124$.

Then, are x and y unique?
No! For example, we can think the following case :

$$\begin{aligned} 15 &= 5 \times 4095 - 124 \times 165 \\ &= 5 \times 4095 - 165 \times 4095 + 165 \times 4095 - 124 \times 165 \\ &= (5 - 165) \times 4095 + (4095 - 124) \times 165 \\ &= -160 \times 4095 + 3971 \times 165 \end{aligned} \tag{6}$$

Here $x = -160$ and $y = 3971$. In conclusion, we can find so many different pairs of x and y. There is no uniqueness for x and y such that $gcd(b, c) = xb + cy$.