1. (a) Use a multiplication table to find all values $a \in \mathbb{Z}_7$ for which the equation

$$x^2 = a$$

has a solution $x \in \mathbb{Z}_7$. For each such a, list all of the solutions x.

(b) Find all solutions $x \in \mathbf{Z}_7$ to the equation $x^2 + \overline{2}x + \overline{6} = \overline{0}$.

2. Consider Z_n.

(a) Under what conditions on n does every nonzero element have a multiplicative inverse? How about an additive inverse?

(b) Does every nonzero element have a multiplicative inverse in Z_{21}?

(c) Does 5 have a multiplicative inverse in Z_{21}? Explain why or why not. If it does, find 5^{-1}.

(d) Solve the equation $5x - 14 = 19$ in $\mathbb{Z}_{21}$.

3. For each of the following, determine if $\sim$ defines an equivalence relation on the set S. If it does, prove it and describe the equivalence classes. If it does not, explain why.

(a) $S = \mathbb{R} \times \mathbb{R}$. For (a, b) and $(c, d) \in S$, define $(a, b) \sim (c, d)$ if $3a + 5b = 3c + 5d$.

(b) $S = \mathbb{R}$. For $a, b \in S$, $a \sim b$ if $a < b$.

(c) $S = \mathbb{Z}$. For $a, b \in S$, $a \sim b$ if $a \mid b$.

(d) $S = \mathbb{R} \times \mathbb{R}$. For (a, b) and $(c, d) \in S$, define $(a, b) \sim (c, d)$ if $\lceil a \rceil = \lceil c \rceil$ and $\lceil b \rceil = \lceil d \rceil$. Here $\lceil x \rceil$ is the smallest integer greater than or equal to x.

4. Use quantifiers to express what it means for a sequence $(x_n)_{n\in\mathbb{N}}$ to *diverge*. You cannot use the terms *not* or *converge*.

5. Suppose $A, B \subseteq \mathbb{R}$ are bounded and non-empty. Show that $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

6. Suppose $S \subseteq \mathbb{R}$ is bounded and non-empty. Define a new set $3S$ by $3S = \{3x \mid x \in S\}$. Show that $\sup(3S) = 3\sup(S)$. Similarly, show that $\inf(3S) = 3\inf(S)$.

7. If $A \subseteq \mathbb{R}$ is bounded above, by the Completeness Axiom, A has a *least upper bound*. Prove that it is unique.

8. Prove that any set $A \subseteq \mathbb{R}$ which is bounded below has a *greatest lower bound*. Furthermore, prove that it is unique.

9. Use the formal definition of limit to prove the following.

   (a) $\lim_{n\to\infty} \frac{n^2+3}{2n^3-4} = 0$

   (b) $\lim_{n\to\infty} \frac{4n-5}{2n+7} = 2$

   (c) $\lim_{n\to\infty} \frac{n^3-3n}{n+5} = +\infty$

   (d) $\lim_{n\to\infty} \frac{n^2-7}{1-n} = -\infty$