Math 299 **Supplement** **Sep 9, 2013**

Counting Sets and Functions

We will learn the basic principles of combinatorial enumeration: counting all possible objects of a specified kind.

The first objects to count are functions whose domain is an interval of integers, $f : \{1, 2, \ldots, n\} \to C$, where C is a given finite set. We will use the notation $[n] = \{1, \ldots, n\}$, so we are dealing with $f : [n] \to C$. These can be formally modeled more neatly than general functions: we can present the data of f simply by listing its values:

$$f = (f(1), f(2), \ldots, f(n)) \in \underbrace{C \times \cdots \times C}_{n \text{ factors}} = C^n.$$

Conversely, any list $(c_1, \ldots, c_n) \in C^n$ represents a function with $f(i) = c_i$ for $i = 1, \ldots, n$. Hence, the number of functions is equal to the number of lists in C^n, namely:

PROPOSITION 1: The number of all possible functions $f : [n] \to C$ is $|C|^n$.

For example, the number of functions $f : [3] \to \{0, 1\}$ is $2^3 = 8$, namely:

$$(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,0),\ (1,0,1),\ (0,1,1),\ (1,1,1).$$

A list like $(1, 0, 1)$ represents the function with $f(1) = 1$, $f(2) = 0$, $f(3) = 1$.

Next, we wish to count all subsets $S \subset [n]$. For example, there are 8 subsets $S \subset [3]$:

$$S = \{\},\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}.$$

Surprisingly, we can reduce this problem to the previous one through the Bijection Principle of combinatorics: if we can transform one data structure into another by an invertible mapping (a bijection), then the two types of data have the same number of possibilities. Formally: suppose we have sets $\mathcal{A}, \mathcal{B}$ and mappings (functions) $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{A}$ which are inverses, meaning that they undo each other:

$$\psi(\phi(a)) = a \ \text{ for all } a \in \mathcal{A}, \ \text{ and } \ \phi(\psi(b)) = b \ \text{ for all } b \in \mathcal{B}.$$

Then ϕ and ψ are bijections, and $|\mathcal{A}| = |\mathcal{B}|$.

In our case, we can define the *Indicator Transform*, an invertible mapping ϕ which changes subsets $S \subset [n]$ into functions $f : [n] \to \{0, 1\}$. That is, if we let $\mathcal{A}$ be the set of all such subsets S, and $\mathcal{B}$ the set of all such functions f, we define a mapping $\phi : \mathcal{A} \to \mathcal{B}$ by:

$$\phi(S) = f, \quad \text{where} \quad f(i) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

We call f the *indicator function* of S. The inverse mapping $\psi : \mathcal{B} \to \mathcal{A}$ takes each function to a corresponding subset:

$$\psi(f) = S = \{i \in [n] \mid f(i) = 1\}.$$

In the example above, each set $S \subset [3]$ has its characteristic function $f : [3] \to \{0, 1\}$ listed in the corresponding place above it in the previous example.

PROPOSITION 2: The number of all possible subsets $S \subset [n]$ is 2^n by Prop. 1.

Proof: If we show that ϕ is a bijection, this will imply $|\mathcal{A}| = |\mathcal{B}|$, and we already know that $|\mathcal{B}|$, the number of functions $f : [n] \to \{0,1\}$, is 2^n.

We check that ϕ, ψ are inverses, using their definitions above:

$$\psi(\phi(S)) = \psi(f) = \{i \mid f(i) = 1\} = \{i \mid i \in S\} = S.$$

Also, $\phi(\psi(f)) = \phi(\{i \mid f(i) = 1\}) = f'$, where $f'(j) = 1$ whenever $j \in \{i \mid f(i) = 1\}$. That is, $f'(j) = 1$ whenever $f(j) = 1$, and otherwise $f'(j) = f(j) = 0$, so $f' = f$ and $\phi(\psi(f)) = f$. This shows that ϕ is invertible, and hence a bijection. Q.E.D.

PROPOSITION 3: (i) The number of possible *injective* functions $f : [n] \to C$ is:

$$|C|\,(|C|{-}1)\,(|C|{-}2)\cdots(|C|{-}n{+}1).$$

(ii) The number of possible bijective functions $f : [n] \to [n]$ is: $n(n{-}1)\cdots(n{-}k{+}1)$.
(iii) The number of possible injective functions $f : [k] \to [n]$ is: $n(n{-}1)\cdots(n{-}k{+}1)$.

Proof. (i) An injective function corresponds to $(f(1), \ldots, f(n))$ with all the entries different from each other. We can choose $f(1)$ to be any element of C, giving $|C|$ possible choices; then for $f(2)$, we can choose any element of C except $f(1)$, giving $|C| - 1$ possibilites; and similarly $|C| - 2$ possibilities for $f(3)$, etc. The number of possible combined choices for f is the product of the individual possibilities, which gives the desired formula.

(ii) From part (i), we see that the number of injective functions $f : [n] \to [n]$ is $n(n{-}1)\cdots(n{-}n{+}1) = n!$. But every injective function is bijective: the image of f has the same size as its domain, namely n, so the image fills the codomain $[n]$, and f is surjective and thus bijective.

(iii) In part (i), replace the domain by $[k]$ and the codomain by $[n]$. Q.E.D.

Our last problem is to count the number of subsets $S \subset [n]$ with a fixed number of elements $|S| = k$. For example, the number of 3-element subsets $S \subset [5]$ is 10:

$$\{1,2,3\},\ \{1,2,4\},\ \{1,2,5\},\ \{1,3,4\},\ \{1,3,5\},\ \{1,4,5\},\ \{2,3,4\},\ \{2,3,5\},\ \{2,4,5\},\ \{3,4,5\}.$$

We define the symbol $\binom{n}{k}$, pronounced "n choose k", to be the answer to the counting problem, so by definition $\binom{5}{3} = 10$. We call these the *choose numbers* or *binomial coefficients.*

PROPOSITION 4: For any integers $0 \le k \le n$, we have:

$$\binom{n}{k} = \frac{n(n{-}1)\cdots(n{-}k{+}1)}{k!}.$$

We will not give a formal proof, but rather examine the above example to see why the formula works. Consider the following table, which contains all the injective functions $f : [3] \to [5]$, each listed in the row corresponding to its image set $S = \{f(1), f(2), f(3)\}$.

$\{1,2,3\}$	$\{1,2,4\}$	$\cdots$	$\{2,4,5\}$	$\{3,4,5\}$
$(1,2,3)$	$(1,2,4)$	$\cdots$	$(2,4,5)$	$(3,4,5)$
$(1,3,2)$	$(1,4,2)$	$\cdots$	$(2,5,4)$	$(3,5,4)$
$(2,1,3)$	$(2,1,4)$	$\cdots$	$(4,2,5)$	$(4,3,5)$
$(2,3,1)$	$(2,4,1)$	$\cdots$	$(4,5,2)$	$(4,5,3)$
$(3,1,2)$	$(4,1,2)$	$\cdots$	$(5,2,4)$	$(5,3,4)$
$(3,2,1)$	$(4,2,1)$	$\cdots$	$(5,4,2)$	$(5,4,3)$

The columns correspond to subsets, which by definition are counted by the unknown value $\binom{5}{3}$. The rows correspond to bijections $g : [3] \to [3]$, and there are $3!$ of these by Prop. 3(ii). The total number of injections in the table is $(5)(4)(3)$ by Prop. 3(iii). Now, the number of columns times the number of rows equals the total number of entries in the table, so we have:

$$\binom{5}{3} \cdot 3! = (5)(4)(3)\,,$$

which immediately gives the desired formula $\binom{5}{3} = \frac{(5)(4)(3)}{3!}$.

In a general proof, we would define an invertible mapping $\phi : \mathcal{S} \times \mathcal{B} \to \mathcal{I}$, where $\mathcal{S}$ is the set of all k-element subsets $S \subset [n]$; $\mathcal{B}$ is the set of all bijections $g : [k] \to [k]$; and $\mathcal{I}$ is the set of all injections $f : [k] \to [n]$. This would guarantee $|\mathcal{S}| \cdot |\mathcal{B}| = |\mathcal{I}|$, that is: $\binom{n}{k} \cdot k! = n(n{-}1)\cdots(n{-}k{+}1)$, giving the desired formula. If you want a challenge, try to define this mapping ϕ and its inverse ψ.