# Math 299    Supplement: Axiomatic Systems    Due: Oct 11, 2013

In informal mathematics, we experiment intuititively with specific numbers, shapes, algorithms, and real-world systems. For each kind of concrete phenomena, we try to capture their key features in a formal Axiomatic System. This is a closed logical universe built on undefined terms, and axioms relating those terms to each other, in which we deduce propositions from the axioms without any appeal to intuition.

We say that the concrete phenomena form a *model* for the Axiomatic System, and the System *axiomatizes* the model.

We give a toy example just to illustrate this paradigm.

## Axiomatic System: "Committee Theory"

**Terms and Axioms.** The undefined terms are: *committee*, *member*, and a member being *in* a committee. We will also use the common logical notions *same* ($=$) and *different* ($\neq$), and basic counting.

These terms are governed by the following assumptions:

- AXIOM 1. Every committee has at least two members in it.

- AXIOM 2. Every member is in at least one committee.

- AXIOM 3. Two committees are the same if the same members are in them.

**Models.** Because this System is so simple, there are many models with the specified features. A model gives a definition to the undefined terms. For example, committees could mean Congressional committees and members could mean congressmen. Or the committees could be some specified sets (or maybe just one set) with at least two elements, and members could be elements of these sets. Or we could take committees to be lines in the plane, and members to be points of those lines.

In any of these models, the Axioms hold true, so any consequences of the Axioms must also hold. The features of the System are only those features common to all possible models. Thus, to prove anything about the System itself, we must work strictly within the System: we cannot appeal to the features of any particular model.

**Definition, Theorem, Proof.** We define new terms by combining undefined terms:

DEFINITION: A member is *popular* means the member is in at least two committees.

THEOREM: If there is a popular member, then there exist at least three members in all.

PROOF: By hypothesis, let $x$ be a popular member. By the definition of popular, $x$ is in at least two different committees $C, D$. By Axiom 1, $C$ has another member $y \neq x$, and $D$ has a member $z \neq x$. If $y \neq z$, then we have the three members desired in the conclusion.

On the other hand, assume $y = z$, so that $x, y$ are in both $C$ and $D$. We know $C, D$ are different committees, so by Axiom 3, they cannot both have only $x, y$ in them. Hence, $C$ or $D$ must have a third distinct member $z$, and the conclusion holds again, Q.E.D.[1]

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[1] *Quod erat demonstrandum*, Latin for "which was to be shown." This often marks the end of a proof.

## Problems

**1.** Read Houston Ch. 16, "How to Read a Theorem," and apply its analysis to the Theorem above.

    **a.** Identify the hypothesis and conclusion.

    **b.** Rate the strength of the hypothesis and conclusion. That is, could we have proved the existence of three members *without* assuming a popular member? Could we have used the hypothesis of a popular member to prove *more* than three members in all? Hint: Use the examples in (c),(d) below.

    **c.** Draw pictures illustrating cases of the Theorem in each of the models mentioned above. (The Theorem becomes pretty irrelevant using the geometry model!)

    **d.** Apply to trivial examples and extreme cases. What does the Theorem look like in the smallest models for the System? and are the hypothesis and conclusion true?

    **e.** Is the converse of the Theorem true?

    **f.** Rewrite the Theorem in symbols, as shortly as possible.

    **g.** Consider some non-examples: models in which the conclusion holds without the hypothesis.

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*The following problems are not mandatory, but rather can be turned in for extra credit.*

    **h.** Generalize by making similar conjectures, or even prove them to make new theorems. The next problem is an example.

**2.** THEOREM: In any committee system satisfying the Axioms, if there exist two popular members, then there exist at least four members in all.

    **a.** Prove this or find a counter-example. Hints: Try to construct a counter-example by playing with the set model. If you can show that a counter-example is impossible, that the hypothesis forces the conclusion, then you have the rough draft of a proof.

    **b.** Try to strengthen this Theorem. That is, do two popular members actually imply at least *five* members in all? Could we get four members assuming a weaker hypothesis than two popular members?