AXIOM 8.1. For all $x, y, z \in \mathbb{R}$,

(i) $x + y \ = \ y + x$

(ii) $(x + y) + z \ = \ x + (y + z)$

(iii) $x \cdot (y + z) \ = \ x \cdot y + x \cdot z$

(iv) $x \cdot y \ = \ y \cdot x$

(v) $(x \cdot y) \cdot z \ = \ x \cdot (y \cdot z)$

AXIOM 8.2. There exists a number 0, such that for all $x \in \mathbb{R}$, $x + 0 \ = \ x$

AXIOM 8.3. There exists a number 1 such that $1 \neq 0$ and whenever $x \in \mathbb{R}$, $x \cdot 1 \ = \ x$

AXIOM 8.4. For each $x \in \mathbb{R}$, there exists a real number, denoted by $-x$, such that $x + (-x) = 0$

AXIOM 8.5. For each $x \in \mathbb{R}\backslash\{0\}$, there exists a real number, denoted by x^{-1}, such that $x \cdot x^{-1} \ = \ 1$

Define **subtraction** in $\mathbb{R}$ by $x - y \ = \ x + (-y)$

AXIOM 8.26. There exists a subset $\mathbb{R}^{>0}$ of $\mathbb{R}$ satisfying

(i) If $x, y \in \mathbb{R}^{>0}$ then $x + y \in \mathbb{R}^{>0}$

(ii) If $x, y \in \mathbb{R}^{>0}$ then $x \cdot y \in \mathbb{R}^{>0}$

(iii) $0 \notin \mathbb{R}^{>0}$

(iv) For every $x \in \mathbb{R}$, we have $x \in \mathbb{R}^{>0}$ or $x = 0$ or $-x \in \mathbb{R}^{>0}$

Members of $\mathbb{R}^{>0}$ are called **positive real numbers**. A negative real number is a real number that is neither positive nor zero.

We write $x < y$ if $y - x \in \mathbb{R}^{>0}$, and say x is **less than** y. Similarly we write $x \leq y$ if $y - x \in \mathbb{R}^{>0}$ or $x = y$, and say x is less than or equal to y.

Let A be a nonempty subset of $\mathbb{R}$. The set A is **bounded above** if there exists $b \in \mathbb{R}$ such that for all $a \in A$, $a \leq b$. Any such number b is called an **upper bound** for A. If b is an upper bound for A that is less than any other upper bound for A, it is called a **least upper bound** for A and is denoted by $sup(A)$ (sup is an abbreviation for supremum).

Note that so far $\mathbb{Q}$ satisfies all the axioms we have listed. For subsets of $\mathbb{Q}$, supremum might not exist within rational numbers such as for $A = \{x \in \mathbb{Q} \mid x^2 < 3\}$. To characterize real numbers we require one more axiom to be satisfied:

AXIOM 8.52. (Completeness axiom). Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

So far we have notations for only two special real numbers: 0 and 1. Next we define $2 = 1 + 1$, $3 = 2 + 1$, ..., $9 = 8 + 1$, which are called digits. Natural numbers within the set of real numbers is defined by finite sums of the form $1 + 1 + ... + 1$, in particular the natural number n corresponds to the sum of n copies of 1.

THEOREM 8.42. $\mathbb{R}^{>0}$ does not have a smallest element.

THEOREM 10.1. The set of natural numbers as a subset of $\mathbb{R}$ is not bounded above in $\mathbb{R}$.

PROPOSITION 10.4. For each $\epsilon \in \mathbb{R}^{>0}$, there exists $n \in \mathbb{N}$ such that $\dfrac{1}{n} < \epsilon$.

PROPOSITION 10.11. Let $x, y \in \mathbb{R}$. Then $x = y$ if and only if for every $\epsilon > 0$ we have $|x - y| \leq \epsilon$.

EXERCISE 1. Using these axioms, show that $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$.

EXERCISE 2. Using these axioms, show that 1 is a positive real number. (Hint: use proof by contradiction, Axiom 8.26(iv) and Axiom 8.26(ii))