1. Let $n \in \mathbb{N}$ and $I = \{1, 2, \ldots, n\}$. For $i \in I$, define $A_i = [(i-1)/n, i/n]$. Identify each of the following sets by writing it as an interval or a union of two intervals.

   (a) $\bigcup_{i \in I} A_i$
   Solution: $\bigcup_{i \in I} A_i = [0, 1]$.

   (b) $\bigcap_{i \in I} \overline{A_i}$
   Solution: $\bigcap_{i \in I} \overline{A_i} = (-\infty, 0) \cup (1, \infty)$.

2. Consider the set $S = \{\emptyset, \square\}$.

   (a) List the elements of $\mathcal{P}(S)$.

   Solution: $\mathcal{P}(S) = \{\emptyset, \{\emptyset\}, \{\square\}, S\}$

   (b) List the elements of $\mathcal{P}(\mathcal{P}(S))$.

   Solution: $\mathcal{P}(\mathcal{P}(S)) = \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\square\}\}, \{S\},\ \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\square\}\}, \{\emptyset, S\}, \{\{\emptyset\}, \{\square\}\},$
   $\{\{\emptyset\}, S\}, \{\{\square\}, S\}, \{\emptyset, \{\emptyset\}, \{\square\}\},\ \{\emptyset, \{\emptyset\}, S\}, \{\emptyset, \{\square\}, S\}, \{\{\emptyset\}, \{\square\}, S\}, \mathcal{P}(S)\Big\}$.

   (c) Find a partition of $\mathcal{P}(\mathcal{P}(S))$ into 3 sets.

   Solution: $A = \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\square\}\}, \{S\},\ \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\square\}\}, \{\emptyset, S\}, \{\{\emptyset\}, \{\square\}\}\Big\}$
   $B = \Big\{\{\{\emptyset\}, S\}, \{\{\square\}, S\}, \{\emptyset, \{\emptyset\}, \{\square\}\}\Big\}$
   $C = \Big\{\{\emptyset, \{\emptyset\}, S\}, \{\emptyset, \{\square\}, S\}, \{\{\emptyset\}, \{\square\}, S\}, \mathcal{P}(S)\Big\}$.

   (d) Is it possible to find a partition of S into 3 sets? Explain.

   Solution: No. A partition into 3 sets would mean that S would need to have 3 non-empty, non-intersecting subsets. This would mean that S would need to have at least 3 *elements, but it only has 2.*

3. Prove the statements appearing in (a)-(b), and answer the prompt in (c)-(d). The symbol $\equiv$ denotes *congruence* modulo n, where $n \in \mathbb{Z}$ such that $n \geq 2$.

   (a) For all $a, b \in \mathbb{Z}$, if $a \equiv b$, then $b \equiv a$.
   Solution: Suppose $a \equiv b$. Then there is some integer k for which $a - b = kn$. This implies $b - a = (-k)n$, and since $-k$ is an integer, it follows that $b \equiv a$.

   (b) For all sets $a, b, c \in \mathbb{Z}$, if $a \equiv b$ and $b \equiv c$, then $a \equiv c$.
   Solution: Suppose $a \equiv b$ and $b \equiv c$. Then there are integers k, l for which $a - b = kn$ and $b - c = ln$. Adding these gives $a - c = (k + l)n$. Since $k + l$ is an integer, it follows that $a \equiv c$.

   (c) State the negation of each of the statements (a)-(b) above. Determine if the negation is true or false. Provide a counterexample for any false statement.
   Solution: The negation of (a) is "There exists $a, b \in \mathbb{Z}$ such that $a \equiv b$, and $b \not\equiv a$." This is false. Then negation of (b) is "There are integers a, b, c such that $a \equiv b$ and $b \equiv c$, but $a \not\equiv c$." This is also false.

   (d) Let $a, b, c \in \mathbb{Z}$, and consider the conditional statement

   P: If $a \equiv b$ and $b \equiv c$, then $a \equiv c$.

   State the inverse, contrapositve and converse of statement P. Determine whether each of these is true or false.
   Solution: The inverse is "If $a \not\equiv b$ or $b \not\equiv c$, then $a \not\equiv c$." The contrapositive is "If $a \not\equiv c$, then $a \not\equiv b$ or $b \not\equiv c$." The converse is "If $a \equiv c$, then $a \equiv b$ and $b \equiv c$." There are values of b for which the inverse and converse is false. The contrapositive is true for all values of a, b, c.

4. Negate the following.

(a) $\forall n \in \mathbb{Z}, \quad \exists m \in \mathbb{Z}$ such that $m \cdot n = 1$.

Solution: The negation is "There is some $n \in \mathbb{Z}$ such that for all $m \in \mathbb{Z}$, $m \cdot n \neq 1$".

(b) $\exists x \in \mathbb{Q}$ such that $\forall y \in \mathbb{Q}, \quad x \cdot y = y$.
Solution: The negation is "For all $x \in \mathbb{Q}$, there is some $y \in \mathbb{Q}$ such that $x \cdot y \neq y$".

Rewrite the statements in (a) and (b) without the use of the symbols $\forall, \exists$, and state whether each is a true or a false statement. If it is a true statement, prove it. If it is a false statement, provide a counterexample.

Solution: Without the logical symbols, (a) and (b) read

(a) For all $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that $m \cdot n = 1$.
(b) There exists some $x \in \mathbb{Q}$ such that for every $y \in \mathbb{Q}, \quad x \cdot y = y$.

(a) is False, but (b) is true. To prove (b), take $x = 1$. Then if $y \in \mathbb{Q}$, we have $x \cdot y = 1 \cdot y = y$.

5. Construct a truth table to show that the contrapositive of $A \Rightarrow B$ is equivalent to $A \Rightarrow B$.

   Solution: I trust you can do this on your own.

6. Prove the following statement.

$$\forall a \in \mathbb{R}\, \exists x \in \mathbb{R}, \text{ such that } 3x - 1 = a.$$

   Solution: Let $a \in \mathbb{R}$. Notice that $\frac{1}{3}(a+1) \in \mathbb{R}$, so letting $x = \frac{1}{3}(a+1)$, we have $3x - 1 = a$.

7. Let E denote the set of even integers, $x \in \mathbb{Z}$, and $A(x)$ be the following open sentence.

$$A(x) : \text{"}x \in E \Rightarrow \exists k \in \mathbb{Z} \text{ such that } x = 2k\text{"}$$

   (a) Write the inverse of $A(x)$.
   Solution: If $x \notin E$, then $\forall k \in \mathbb{Z}, x \neq 2k$.
   (b) Write the converse of $A(x)$.
   Solution: If $\exists k \in \mathbb{Z}$ such that $x = 2k$, then $x \in E$.
   (c) Write the contrapositive of $A(x)$.
   Solution: If $\forall k \in \mathbb{Z}, x \neq 2k$, then $x \notin E$.
   (d) Is $A(x)$ true for all $x \in \mathbb{Z}$? What about its converse? In this case, how would you restate it using *necessary/ sufficient/ necessary and sufficient*?
   Solution: The statement $A(x)$, and its converse are both true for all $x \in E$. Combining these, they can be phrased as saying "being able to write $x = 2k$ for some integer k is both a necessary and sufficient condition for x to be even."

8. Let $A = \{x \in \mathbb{Z} | x = 6k, k \in \mathbb{Z}\}$, $B = \{x \in \mathbb{Z} | x = 2k, k \in \mathbb{Z}\}$, $C = \{x \in \mathbb{Z} | x = 3k, k \in \mathbb{Z}\}$. Prove the following statement.

$$x \in A \iff (\exists y \in B \text{ and } \exists z \in C \text{ such that } x = yz)$$

   Solution: Suppose $x \in A$, so we can write $x = 6k$ for an integer k. Then $x = (2k)(3l)$, where $l = 1$. Since $2k \in B$ and $3l \in C$ it follows that $x = yz$ for some $y \in B$ and $z \in C$.

   Conversely, suppose $x = yz$ for some $y = 2k \in B$ and $z = 3l \in C$. Then $x = (2k)(3l) = 6(lk)$. Since lk is an integer, it follows that $x \in A$.

9. Construct a truth table to show that $A \Rightarrow B$ is equivalent to the statement: not(A) or B.

   Solution: You are on your own for this one.

10. Let $a, b, c \in \mathbb{R}$, and consider the following open sentence:

$P(a,b,c)$: A necessary condition for the equation $ax^2+bx+c=0$ to have a solution is: $a \neq 0$ and $b^2-4ac \geq 0$.

(a) Rephrase $P(a,b,c)$ as an if-then implication; explicitly write all relevant quantifiers.
Solution: If there exists $x \in \mathbb{R}$ such that $ax^2+bx+c=0$, then $a \neq 0$ and $b^2-4ac \geq 0$.

(b) Write the contrapositive.
Solution: If $a=0$ or $b^2-4ac<0$ then for all $x \in \mathbb{R}$, we have $ax^2+bx+c \neq 0$.

(c) Write the converse.
Solution: If $a \neq 0$ and $b^2-4ac \geq 0$, then there exists $x \in \mathbb{R}$ such that $ax^2+bx+c=0$.

(d) Write the inverse.
Solution: If for all $x \in \mathbb{R}$, we have $ax^2+bx+c \neq 0$, then $a=0$ or $b^2-4ac<0$.

(e) Write the negation of $P(a,b,c)$ (simplified by moving the *not* as far into the statement as possible).
Solution: The negation is "there exists $x \in \mathbb{R}$ such that $ax^2+bx+c=0$, and ($a=0$ or $b^2-4ac<0$).

(f) Which of the above statements (a)-(d) are equivalent to each other?
Solution: The statements in (a) and (b) are equivalent (for all a,b,c), and the statements (c) and (d) are equivalent to each other. However, the statements (a) and (b) are not equivalent to the statements (c) and (d) (there are some values of a,b,c for which (a) and (b) are false, but the converse is always true).

(g) The statement 'for all $a,b,c \in \mathbb{R}, P(a,b,c)$' is false. Disprove it (prove the negation).
Solution: The negation is $\exists a,b,c \in \mathbb{R}$ such that $P(a,b,c)$ is not true. Indeed, let $a=0, b=1$ and $c=0$. Then there is some $x \in \mathbb{R}$ (namely $x=0$) with $0=ax^2+bx+c$.

11. Let $x,y \in \mathbb{R}$. Prove that $(x+y)^2=x^2+y^2$ if and only if $xy=0$.

Solution: Notice that the statement $(x+y)^2=x^2+y^2$ is true if and only if $x^2+2xy+y^2=x^2+y^2$ is true. Subtracting x^2+y^2 shows that this latter statement is true if and only if $2xy=0$, which is equivalent to $xy=0$.

12. Let $n \in \mathbb{Z}$. Prove that n is odd if and only if $n+7$ is even.

Solution: Assume $n=2k+1$ is odd. Then $n+7=2k+8=2(k+4)$ is even. Conversely, suppose $n=2k$ is even. Then $n+7=2k+7=2(k+3)+1$ is odd.

13. Let $n \in \mathbb{Z}$. Prove that n is odd if and only if n^2 is odd.

Solution: If $n=2k+1$ is odd, then $n^2=(2k+1)^2=2(2k^2+2k)+1$ is odd. Conversely, if $n=2k$ is even, then $n^2=(2k)^2=2(2k^2)$ is even.

14. Let $a \in (0,\infty)$. Prove that a rectangle with perimeter $4a$ is a square if and only if its area is a^2.

Solution: Let $a \in (0,\infty)$, and suppose R is a rectangle with perimeter $4a$. Let x,y denote the lengths of the sides of R. If R is a square, then $x=y$, so the perimeter is $2x+2y=4x$. This implies that $a=x$, and so the area is $xy=x^2=a^2$. Conversely, suppose the area is a^2. The area is also given by xy. So we have two equations:

$$2x+2y=4a, xy=a^2.$$

The first gives $x+y=2a$, and squaring gives $x^2+2xy+y^2=4a^2$. Now combine this with the second equation to write $x^2+2xy+y^2=4xy$, and so

$$x^2-2xy+y^2=0.$$

The left-hand side factors to give $(x-y)^2=0$, and so $x=y$, which implies R is a square.

15. Define the *Euclidean norm* of $\mathbf{x}=(x_1,x_2,...,x_n) \in \mathbb{R}^n$ by $||\mathbf{x}||=\sqrt{x_1^2+...+x_n^2}$. Prove that $||\mathbf{x}||=0$ if and only if $(x_1,x_2,...,x_n)=(0,0,...,0)$.

Solution: Suppose $(x_1,\ldots,x_n)=(0,\ldots,0)$. Then $\|\mathbf{x}\|=\sqrt{0^2+\ldots+0^2}=0$. To prove the converse, assume $\|\mathbf{x}\|=0$. Notice that since $x_j^2 \geq 0$ for all j, we have $x_j^2 \leq x_1^2+\ldots+x_n^2$ for every j. Taking the square root of both sides gives

$$|x_j|=\sqrt{x_j^2} \leq \|\mathbf{x}\|$$

(this uses the fact that the square root is an increasing function, and so preserves the inequality). The right-hand side is zero by assumption, and so $|x_j| \leq 0$*. This implies* $x_j = 0$ *for all* j*.*

16. Let $x \in \mathbb{Z}$.

    (a) Prove that $x^2 + x$ is even.

    Solution: If $x = 2k$ *is even, then* $x^2 + x = 2(2k^2 + k)$ *is even. If* $x = 2k + 1$ *is odd, then* $x^2 + x = 4k^2 + 4k + 1 + 2k + 1 = 2(2k^2 + 2k + 1)$ *is even again.*

    (b) Assume $x \neq 0$. Prove that $(x^2 + x)/2$ is divisible by x if and only if x is odd.

    Solution: Write $(x^2 + x)/2 = x(x + 1)/2$*. If* x *is odd, then* $x + 1$ *is even, and so* $(x + 1)/2 = k$ *is an integer. It follows that* $x(x + 1)/2 = xk$ *is divisible by* x*. Conversely, if* $x(x + 1)/2$ *is divisible by* x*, then write* $x(x + 1)/2 = kx$ *for some integer* k*. As I mentioned, we need to assume* $x \neq 0$*. Then we can divide by* x *to get* $(x + 1)/2 = k$*. If follows that* $x = 2k - 1$ *is odd.*

    (c) Assume $x + 1 \neq 0$. Prove that $(x^2 + x)/2$ is divisible by $x + 1$ if and only if x is even.

    Solution: The proof is essentially the same as the previous part.

17. Show that if $x^2 - 3x + 2 < 0$, then $1 < x < 2$.

    Solution: First note that the polynomial factors as $x^2 - 3x + 2 = (x - 1)(x - 2)$*. We prove the contrapositive: If* $x \geq 2$*, then* $x \geq 1$ *and so* $(x - 1)$ *and* $(x - 2)$ *are both greater than or equal to zero. It follows that* $x^2 - 3x + 2 \geq 0$*. On the other hand, if* $x \leq 1$*, then* $x \leq 2$ *as well, so* $(x - 1)$ *and* $(x - 2)$ *are both less than or equal to zero. It follows that* $x^2 - 3x + 2 \geq 0$ *in this case as well.*

18. Let $a, b, c, d \in \mathbb{Z}$ with a and b nonzero. Prove that if $ab \nmid cd$, then $a \nmid c$ or $b \nmid d$.

    Solution: We will prove the contrapositive, so assume $a|c$ *and* $b|d$*. Then write* $c = ak$ *and* $d = bl$ *for integers* k, l*. This gives* $cd = ab(kl)$*, which shows* $ab|cd$*.*

19. Prove that for any two sets A and B, $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$.

    Solution: Let $x \in \overline{(A \cup B)}$*. Then* x *is not in* $A \cup B$*. This means that* $x \notin A$ *and* $x \notin B$ *(because otherwise* x *would belong to the union). Therefore,* $x \in \overline{A}$ *and* $x \in \overline{B}$*. Hence,* $x \in \overline{A} \cap \overline{B}$*. This proves that* $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$*.*

    Next, let $x \in \overline{A} \cap \overline{B}$*. Then* $x \in \overline{A}$ *and* $x \in \overline{B}$*. Therefore,* x *does not belong to* A *and* x *does not belong to* B*. Hence,* x *is not an element of* $A \cup B$ *(because otherwise* x *would belong to* A *or to* B*). This proves that that* $x \in \overline{A \cup B}$ *and completes the proof that* $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$*.*

    Since we have shown that the two given sets are subsets of each other, they contain the same elments and hence are equal.

20. Prove that for any two sets A and B, $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

21. Prove that for any sets A, B and C, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

22. Prove that if $n|a$ then $n|a + b \Leftrightarrow n|b$

    Solution: Assume $n|a$ *and* $n|a + b$*. Then write* $a = nk$ *and* $a + b = nl$ *for integers* k, l*. Then* $b = nl - a = n(l - k)$*, which shows* $n|b$*.*