1. Describe the elements of the set $(\mathbb{Z} \times \mathbb{Q}) \cap \mathbb{R} \times \mathbb{N}$. Is this set countable or uncountable?

2. Let $A = \{\emptyset, \{\emptyset\}\}$. What is the cardinality of A? Is $\emptyset \subset A$? Is $\emptyset \in A$? Is $\{\emptyset\} \subset A$? Is $\{\emptyset\} \in A$? Is $\{\emptyset, \{\emptyset\}\} \in A$?

3. List the elements of the set $A \times B$ where A is the set in the previous question and $B = \{1, 2\}$.

4. Suppose that A, B, and C are sets. Which of the following statements is true for all sets A, B, and C? For each, either prove the statement or give a counterexample: $(A \cap B) \cup C = A \cap (B \cup C)$, $A \cap B \subseteq A \cup B$, if $A \subset B$ then $A \times A \subset A \times B$, $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$.

5. State the negation of each of the following statements:
   - There exists a natural number m such that $m^3 - m$ is not divisible by 3.
   - $\sqrt{3}$ is a rational number.
   - 1 is a negative integer.
   - 57 is a prime number.

6. Verify the following laws:
   - (a) Let P, Q and R are statements. Then,

     $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are logically equivalent.
   - (b) Let P and Q are statements. Then,

     $P \Rightarrow Q$ and $(\sim Q) \Rightarrow (\sim P)$ are logically equivalent.

7. Write the open statement $P(x, y)$: **"for all real x and y the value $(x-1)^2 + (y-3)^2$ is positive"** using quantifiers. Is the quantified statement true or false? Explain.

8. Prove that $3x + 7$ is odd if and only if x is even.

9. Prove that if a and b are positive numbers, the $\sqrt{ab} \leq \frac{a+b}{2}$. This is referred to as "Inequality between geometric and arithmetic mean."

10. Let A, B, and C be sets. Prove that $A \times (B \bigcap C) = (A \times B) \bigcap (A \times C)$.

11. Let A, B, and C be sets. Prove that $(A - B) \bigcap (A - C) = A - (B \bigcup C)$.

12. Suppose that x and y are real numbers. Prove that if $x + y$ is irrational, then x is irrational or y is irrational.

13. Let x be an irrational number. Prove that x^4 or x^5 is irrational.

14. Use a proof by contradiction to prove the following.

    There exist no natural numbers m such that $m^2 + m + 3$ is divisible by 4.

15. Let a, b be distinct primes. Then $\log_a(b)$ is irrational.

16. Prove or disprove the statement: there exists an integer n such that $n^2 - 3 = 2n$.

17. Prove or disprove the statement: there exists a real number x such that $x^4 + 2 = 2x^2$.

18. Prove that there exists a unique real number x such that $x^3 + 2x = 2$.

19. Disprove that statement: There exists integers a and b such that $a^2 + b^2 \equiv 3 \pmod 4$

20. Use induction to prove that $6|(n^3 + 5n)$ for all $n \geq 0$.

21. Use induction to prove that

    $$1 \cdot 4 + 2 \cdot 7 + \cdots + n(3n+1) = n(n+1)^2$$

    for all $n \in \mathbb{N}$.

22. Use the Strong Principle of Mathematical Induction to prove that for each integer $n \geq 13$, there are nonnegative integers x and y such that $n = 4x + 5y$.

23. A sequence $\{a_n\}$ is defined recursively by $a_0 = 1$, $a_1 = -2$ and for $n \geq 1$,

    $$a_{n+1} = 5a_n - 6a_{n-1}.$$

    Prove that for $n \geq 0$,

    $$a_n = 5 \times 2^n - 4 \times 3^n.$$

24. Suppose R is an equivalence relation on a set A. Prove or disprove that R^{-1} is an equivalence relation on A.

25. Consider the set $A = \{a, b, c, d\}$, and suppose R is an equivalence relation on A. If R contains the elements (a, b) and (b, d), what other elements must it contain?

26. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$. Find a relation on $A \times B$ that is transitive and symmetric, but not reflexive.

27. Suppose A is a finite set and R is an equivalence relation on A.

(a) Prove that $|A| \leq |R|$.

(b) If $|A| = |R|$, what can you conclude about R?

28. Consider the relation $R \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by

$$R = \{(x \bmod 4, 3x \bmod 6) \mid x \in \mathbb{Z}\}.$$

Prove that R is a function from $\mathbb{Z}_4$ to $\mathbb{Z}_6$. Is R a bijective function?

29. Consider the relation $S \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by

$$S = \{(x \bmod 4, 2x \bmod 6) \mid x \in \mathbb{Z}\}.$$

Prove that S is not a function from $\mathbb{Z}_4$ to $\mathbb{Z}_6$.

30. Suppose $f : A \to B$ and $g : X \to Y$ are bijective functions. Define a new function $h : A \times X \to B \times Y$ by $h(a, x) = (f(a), g(x))$. Prove that h is bijective.

31. Prove or disprove: Suppose $f : A \to B$ and $g : B \to C$ are functions. Then $g \circ f$ is bijective if and only if f is injective and g is surjective.

32. *(X points)* Let $\mathbb{R}^+$ denote the set of positive real numbers and let A and B be denumerable subsets of $\mathbb{R}^+$. Define $C = \{x \in \mathbb{R} : -x/2 \in B\}$. Show that $A \cup C$ is denumerable.

33. Prove that the interval $(0, 1)$ is numerically equivalent to the interval $(0, +\infty)$.

34. Prove the following statement: A nonempty set S is **countable** if and only if there exists an injective function $g : S \to \mathbb{N}$.

35. Consider the set $S = \left\{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\right\}$. Prove that $\mathbb{R} - S$ is uncountable.

36. (a) Suppose A, B are sets. Prove that if A and B have the same cardinality, then $A \times \mathbb{Z}$ and $B \times \mathbb{Z}$ have the same cardinality.

(b) Prove that $\mathbb{Z}^n$ has the same cardinality as $\mathbb{Z}^{n+1}$ for all $n \in \mathbb{N}$. *Hint: Induct on n, and use part (a) for the inductive step.*

(c) Prove that $\mathbb{Z}^n$ is countable for all $n \in \mathbb{N}$.

37. Compute the greatest common divisor of 42 and 13 and then express the greatest common divisor as a linear combination of 42 and 13.

38. Let $a, b, c \in \mathbb{Z}$. Prove that if c is a common divisor of a and b, then c divides any linear combination of a and b.

39. Define the term "p is a prime". Then prove that if $a, p \in \mathbb{Z}$, p is prime, and p does not divide a, then $\gcd(a, p) = 1$.

40. The greatest common divisor of three integers a, b, c is the largest positive integer which divides all three. We denote this greatest common divisor by $\gcd(a, b, c)$. Assume that a and b are not both zero. Prove the following equation:

$$\gcd(a, b, c) = \gcd(\gcd(a, b), c).$$

41. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove the following:

$$\lim_{n\to\infty} \frac{3n + 1}{n - 2} = 3.$$

42. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove that

$$\lim_{n\to\infty} \frac{(-1)^n 3n + 1}{n - 2}$$

does not exist.

43. Let (a_n) be a sequence with positive terms such that $\lim_{n\to\infty} a_n = 1$. By using the formal definition of the limit of the sequence, prove the following:

$$\lim_{n\to\infty} \frac{3a_n + 1}{2} = 2.$$

44. (a) Use induction to prove

$$\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \cdots + \frac{1}{2n(2n + 2)} = \frac{n}{4(n + 1)}$$

for all $n \in \mathbb{N}$.

(b) Prove $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k(2k + 2)} = \frac{1}{4}$.