Math 299 **Lecture 11: Section 4.3**
Proofs Involving Real Numbers

Properties:

A1 For all real numbers a, b, c, if $a \leq b$ and $b \leq c$ then $a \leq c$.

A2 For all real numbers a, b, c, if $a \leq b$ then $a + c \leq b + c$.

A3 For all real numbers a, b, c, if $a \leq b$ and $0 \leq c$ then $ac \leq bc$.

Prove the statements below using A1-A3, together with any basic facts about *equality* $=$.

1. For all real numbers a, if $a \leq 0$ then $0 \leq -a$.
2. For all real numbers a, $a^2 \geq 0$.
3. For all real numbers a, b, $ab \leq \frac{1}{2}(a^2 + b^2)$. *Hint: Consider* $(a - b)^2$.
4. For all real numbers a, b, δ, if $\delta \neq 0$ then $ab \leq \frac{1}{2}(\delta^2 a^2 + \delta^{-2} b^2)$.
5. For all real numbers a, b, $ab = \frac{1}{2}(a^2 + b^2)$ if and only if $a = b$.

Working Backwards

Inequality between arithmetic and geometric mean.
If $a, b \in \mathbb{R}$ with $a \geq 0$ and $b \geq 0$, then $\dfrac{a+b}{2} \geq \sqrt{ab}$.

Scratch work:

1. Start with the inequality you are asked to prove.
2. Simplify it as much as possible until you arrive at a statement that is obviously true.

Formal Proof:

3. In order to write formal proof, now start from the obviously true statement.
4. Use your previous work to guide you on how to arrive at the desired inequality.

Proving Equalities

Theorem: Let $x, y \in \mathbb{R}$. Then $xy = 0$ if and only if $x = 0$ or $y = 0$.

Prove the following:

1. Let $x \in \mathbb{R}$. If $x^3 - 3x^2 + x = 3$ then $x = 3$.

2. Let $x, y \in \mathbb{R}$, then $\frac{5}{6}x^2 + \frac{3}{10}y^2 \geq xy$.

Proving Inequalities

Triangle Inequality. Let $x, y \in \mathbb{R}$. Then $|x+y| \leq |x| + |y|$.

Prove the following:

Let $x \in \mathbb{R}$, then $||x| - |y|| \leq |x - y|$.

EX. Using a "Triangle Inequality", prove the following implication.

If $|x-1| < 1$ and $|x-1| < r/4$ for $r > 0$ and $r \in \mathbb{R}$, then $|(x+2)(x-1)| < r$.