Math 299 **Lectures 17 and 18: Sections 6.1 - 6.2**
Mathematical Induction.

Definition: A nonempty set S of real numbers is **well-ordered** if every nonempty subset of S has a least element.

- Example of a well-ordered set:

- Example of a set which is not well-ordered:

The Well-Ordering Principle: The set $\mathbb{N}$ is well-ordered.

Principle of Mathematical Induction

For each natural number n, let $P(n)$ be a statement. We like to demonstrate that $P(n)$ is true for all $n \in \mathbb{N}$.

To show that $P(n)$ holds for all natural numbers n, it suffices to establish the following:

I. Base case: Show that $P(1)$ is true.

II. Induction step:

(i) Assume that $P(k)$ holds for an arbitrary $k \in \mathbb{N}$.
This step is "Induction hypothesis".

(ii) Show that "$P(k)$ is true" (a hypothesis) implies "$P(k+1)$ is also true" (a conclusion).

III. Conclusion: P(n) is true for all $n \in \mathbb{N}$

This is sometimes referred to as the domino effect. Once one of the dominoes topples it causes the rest to topple as well.

$$P(1) \implies P(2),\ P(2) \implies P(3), \cdots.\ P(k) \implies P(k+1).$$

The Principle of Mathematical Induction

Theorem: For each $n \in \mathbb{N}$, let $P(n)$ be a statement. If

1. $P(1)$ is true and
2. the implication **If P(k), then P(k+1).** is true $\forall k \in \mathbb{N}$,

then $P(n)$ is true $\forall n \in \mathbb{N}$

Prove this theorem using proof by contradiction, and using the fact that $\mathbb{N}$ is well-ordered.

Where is the mistake in the proof or are indeed all horses the same color?

All Horses Have the Same Color

I. Base case: One horse

The case with just one horse is trivial. If there is only one horse in the "group", then clearly all horses in that group have the same color.

II. Induction step:

Assume that n horses always are the same color. Let us consider a group consisting of n+1 horses.

First, exclude the last horse and look only at the first n horses; all these are the same color since n horses always are the same color.

Likewise, exclude the first horse and look only at the last n horses. These too, must also be of the same color.

Therefore, the first horse in the group is of the same color as the horses in the middle, who in turn are of the same color as the last horse. Hence the first horse, middle horses, and last horse are all of the same color, and we have proven that:

If n horses have the same color, then n+1 horses will also have the same color.

III. Conclusion:

We already saw in the base case that the rule ("all horses have the same color") was valid for n=1.

The inductive step showed that since the rule is valid for n=1, it must also be valid for n=2, which in turn implies that the rule is valid for n=3 and so on.

Thus in any group of horses, all horses must be the same color.

Examples

I. Base case: Show that $P(1)$ is true.

II. Induction step :

  (i) Assume that $P(k)$ holds for an arbitrary $k \in \mathbb{N}$, $k \geq i$.

  (ii) Show that "$P(k)$ is true" implies "$P(k+1)$ is also true."

III. Conclusion.

Prove that $\sum_{x=1}^{n} x = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$.

1. What is the statement $P(n)$ proven to be true by the induction method?

2. What does this statement mean? Check $n = 2, 3, 4, 5$ so on.

3. How can we show that the statement $P(n)$ is true for all natural number n?

4. What is the base case?

5. In an induction step, what is the hypothesis and the conclusion?

6. Prove $P(n)$ is true for all natural number n.

Exercises

I. Base case: Show that $P(1)$ is true.

II. Induction step :

  (i) Assume that $P(k)$ holds for an arbitrary $k \in \mathbb{N}$, $k \geq i$.

  (ii) Show that "$P(k)$ is true" implies "$P(k+1)$ is also true."

III. Conclusion.

1. Prove for all integers $n \in \mathbb{N}$

$$1 \times 1! + 2 \times 2! + \cdots n \times n! = (n+1)! - 1.$$

2. Let $x_1 = 1$ and for each $n \in \mathbb{N}$ define $x_{n+1} = \frac{1}{2}x_n + 1$. Prove that $x_n \leq 2$ for every $n \in \mathbb{N}$.

3. Prove that $n^3 - n$ is divisible by 3 $\forall n \in \mathbb{N}$.

More general induction

Theorem: For a fixed integer m, let $S = \{i \in \mathbb{Z} : i \geq m\}$. For each $n \in S$, let $P(n)$ be a statement. If

1. $P(m)$ is true and
2. the implication **If P(k), then P(k+1).** is true $\forall k \in S$,

then $P(n)$ is true $\forall n \in S$.

Example. Prove that $n! \geq n^2$ $\forall n \in \mathbb{N}, n \geq 4$.

Example. $\sum_k^n \frac{1}{k^2} < 2 - \frac{1}{n}$ for all integers $n \geq 2$.

More exercise problems Prove each statement by a using induction method

1. $\sum_{x=1}^{n} \frac{1}{\sqrt{x}} \leq 2\sqrt{n}$.

2. $(2^{2n-1}+1)$ is divisible by 3 $\forall n \in \mathbb{N}$.

3. $12^n - 5^n$ is divisible by 7 for every nonnegative integer n.

4. The sum of cubes of three consecutive natural numbers is divisible by 9.

5. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$.

6. Let a and b be real numbers. Then for natural numbers $n \geq 2$,

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}\right)$$

7. If $S_1, ..., S_n$ are $n \geq 2$ sets, then $\overline{\bigcap_{i=1}^{n} S_i} = \bigcup_{i=1}^{n} \overline{S_i}$.

8. (Challenging) Let $g(x) = \mathrm{e}^{-1/x}$. For $x > 0$, $n \geq 0$, the n^{th} derivative of g is

$$g^{(n)}(x) = P\left(\frac{1}{x}\right)\mathrm{e}^{-1/x},$$

where P is a polynomial function.