Math 299 **Lecture 29: Sections 10.1-10.3**
Cardinalities of Sets

Numerically equivalent sets

- If two sets A and B are both empty, A and B have the same cardinality.
- Two **finite** sets have the same number of elements, they have the same cardinality. Such sets are referred to as **numerically equivalent sets**.
- What do we mean if we say that two **infinite** sets are numerically equivalent sets (have the same cardinality)?
  What if we can find a bijective function f between two infinite sets?

Definition: Two sets A and B are said to have the **same cardinality**, that is, $|A| = |B|$ if there exists a **bijective** function f from A to B.

Definition: A set A is called **denumerable** if A has the same cardinality as the set of natural number.

1. An *infinite* set X is **countably infinite** (or **countable**)
   if there exists a bijection between X and $\mathbb{N}$.

   - Ex : $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}$ so on.

2. An *infinite* set X is **uncountably infinite**
   if X has more than countably many elements.

   - Ex : $\mathbb{R}$, the set of real numbers from 0 to 1 (i.e, $[0, 1]$).
     $\mathbb{R} - \mathbb{Q}$ (the set of irrational numbers) so on.

Examples of countable sets

1. Prove that the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is countable.

2. Prove that the set of odd (positive) numbers is countable.

3. Prove that the set of $\mathbb{Z}$ is countably infinite.

Theorem: Every infinite subset of a denumerable set is denumerable.

4. Prove that the rationals, $\mathbb{Q}$, are countable

First, let us write all possibilities for a/b in a grid as follows:

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Examples of uncountable sets

- The set of real numbers, $\mathbb{R}$, is uncountable.
- Every nonempty interval (x, y) is uncountable.

The set of real numbers in the interval $[0, 1]$ *is uncountable*

1. Assumption : there exists a bijection between $\mathbb{N}$ and $[0, 1]$.
2. By assumption, we can list all real numbers in $[0, 1]$.
3. Show that there is a new real number in $[0, 1]$ which is not on the list.

Cantor's argument : $|\mathbb{R}| > |\mathbb{N}|$ **(An uncountable set)**

Cantor's argument was a mathematical proof that there exist infinite sets which don't have a one–to–one correspondence with the infinite set of natural numbers. Now such sets are called uncountable sets.

To prove $|\mathbb{R}| > |\mathbb{N}|$, we assume that the set of $\mathbb{R}$ is countable. Let us consider a subset [0,1] of $\mathbb{R}$. By the assumption, we can list all real numbers in $[0, 1]$.

$$\begin{array}{lcccccccc}
1: & 0 & . & \color{red}{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\
2: & 0 & . & a_{21} & \color{red}{a_{22}} & a_{23} & a_{24} & a_{25} & \cdots \\
3: & 0 & . & a_{31} & a_{32} & \color{red}{a_{33}} & a_{34} & a_{35} & \cdots \\
4: & 0 & . & a_{41} & a_{42} & a_{43} & \color{red}{a_{44}} & a_{45} & \cdots \\
5: & 0 & . & a_{51} & a_{52} & a_{53} & a_{54} & \color{red}{a_{55}} & \cdots \\
\vdots & & & & & \vdots & & & \ddots
\end{array}$$

where $a_{ij} \in \{0, 1, 2, \cdots, 7, 8, 9\}$. Construct a new real number x which is in $[0, 1]$, but not in the above list.

Corollary: $\mathbb{R}$ is uncountable.

How can we prove this?

Interesting facts

- $|\mathbb{R}|$ is referred to as **cardinality of the continuum**
- $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$
- Cantor's Theorem: $|\mathcal{P}(A)| > |A|$ for any set A.
- Continuum Hypothesis: $\nexists A$ such that $|\mathbb{N}| < |A| < |\mathbb{R}|$.

  The Cotinuum Hypothesis is independent of the axioms of Set Theory. That is, both the hypothesis and its negation are consistent with these axioms.