
Definitions.
Let S be a nonempty subset of $\mathbb{R}$, i.e. $\phi \neq S \subseteq \mathbb{R}$

(1) If $x_0 \in S$ and $x \leq x_0$ for all $x \in S$,
then x_0 is called the maximum of S. ($x_0 = \max S$.)

(2) If $x_0 \in S$ and $x_0 \leq x$ for all $x \in S$,
then x_0 is called the minimum of S. ($x_0 = \min S$.)

(3) If $\exists M \in \mathbb{R}$ such that $x \leq M$ for all $x \in S$,
then M is called an **upper bound** of S and the set S is **bounded above**.

(4) If $\exists m \in \mathbb{R}$ such that $m \leq x$ for all $x \in S$,
then m is called a **lower bound** of S and the set S is **bounded below**.

(5) If $\exists m, M \in \mathbb{R}$ such that $m \leq x \leq M$ $\forall x \in S$, then S is **bounded**.

(6) If S is bounded above and S has a least upper bound M_0, then M_0 is called the supremum of S and denoted by $\sup S$.

(7) If S is bounded below and S has a greatest lower bound m_0, then m_0 is called the infimum of S and denoted by $\inf S$.

The Completeness Axiom. A fundamental property of the set $\mathbb{R}$ of real numbers is that $\mathbb{R}$ has "no gaps", i.e.,

$\forall S \subseteq \mathbb{R}$ and $S \neq \emptyset$, if S is bounded above, then $\sup S$ exists and $\sup S \in \mathbb{R}$.

(that is, the set S has a least upper bound which is a real number).

Note: The Completeness Axiom distinguishes the set of real numbers $\mathbb{R}$ from the set of rational numbers $\mathbb{Q}$.

– EX: Let $A := \{r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2}\} \subseteq \mathbb{Q}$.

(1) Is the set A bounded above?

(2) Does it has a least upper bound in A?

Examples.

Find the inf and sup of the following sets, if possible. State whether or not these numbers are in S.

1. $S = \{x \,|\, 0 < x \leq 3\}$

2. $S = \{x \,|\, x^2 - 2x - 3 < 0\}$

3. $S = \{x \,|\, 0 < x < 5, \cos(x) = 0\}$

4. $S = \{x \,|\, x = \frac{1}{n}, n \in \mathbb{N}\}$

Some properties of sup and inf

Theorem. If x_1 and x_2 are least upper bounds for the set A, then $x_1 = x_2$.

Theorem. If the sets A and B are bounded above and $A \subseteq B$, then $\sup(A) \leq \sup(B)$.

Chapter 12.1: Limits of Sequences

Definition: A **sequence** in a set S is a function from $\mathbb{N}$ to S.

Definition (Limit of a sequence):
If, $\forall\ \varepsilon > 0$, $\exists N = N(\varepsilon)$ such that $\forall n > N$, $\mid x_n - x \mid \leq \varepsilon$, then a sequence (x_n) of real numbers **converges** to the real number x.

(We write $\lim_{n\to\infty} x_n = x$, "$x$" is the limit of the sequence (x_n).)

Definition: If a sequence (x_n) does not converge to some real number, then the sequence (x_n) diverges.

Write the negation of convergence using quantifiers.

Examples

1. Prove that $\lim_{n\to\infty} \dfrac{1}{n} = 0$.

2. Prove that $\lim_{n\to\infty} 1 = 1$.

3. Prove that $\lim_{n\to\infty} \frac{3}{2n+1} = 0$.

4. Prove that $\lim_{n\to\infty} \frac{2n+1}{n+1} = 2$.

5. Prove that the sequence $a_n = 1 + (-1)^n$ is divergent.

Examples

1. Prove that $\lim_{n\to\infty} \frac{n-2}{2n+1} = \frac{1}{2}$.

2. Prove that $\lim_{n\to\infty} \frac{n+1}{n^2} = 0$.

3. Prove that $\lim_{n\to\infty} \frac{2n}{n^2+3} = 0$.

4. Prove that $\lim_{n\to\infty} \frac{2n}{n^2-3} = 0$.

5. Prove that $\lim_{n\to\infty} \frac{n^2 + 2n}{n^3 - 5} = 0$.

Some Properties of Real Numbers
Prove the following.

Proposition. Let x, $y \in \mathbb{R}$. Then $x = y$ if and only if $\forall \varepsilon > 0$ we have $|x - y| \leq \varepsilon$.

Some properties of limit.

Theorem 1. If a sequence (a_n) converges, then its limit is unique.

Theorem 2. Every convergent sequence must be bounded.

Theorem 3. Algebraic rules for sequences:
Let $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$.

(a) For $k \in \mathbb{R}$, $\lim_{n\to\infty} ks_n = k \lim_{n\to\infty} s_n = ks$.

(b) $\lim_{n\to\infty} (s_n + t_n) = s + t$.

(c) $\lim_{n\to\infty} (s_n \cdot t_n) = s \cdot t$.

(d) For all n, $s_n \neq 0$ and $s \neq 0$, $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ and $\lim_{n\to\infty} \frac{t_n}{s_n} = \frac{t}{s}$.

Divergence
Definition

(1) If $\forall M > 0$, $\exists N$ such that $\forall n > N$, $n \in \mathbb{N}$, $s_n > M$, then the sequence diverges to $+\infty$. We write $\lim_{n\to\infty} s_n = +\infty$.

(2) If $\forall M < 0$, $\exists N$ such that $\forall n > N$, $n \in \mathbb{N}$, $s_n < M$, then the sequence diverges to $-\infty$. We write $\lim_{n\to\infty} s_n = -\infty$.

Examples

1. Give a formal proof that $\lim_{n\to\infty} (\sqrt{n} + 7) = +\infty$.

2. Prove that $\lim_{n\to\infty} \dfrac{n^2 + 4}{n + 2} = +\infty$.

3. Prove that $\lim_{n\to\infty} \dfrac{n^3}{1 - n} = -\infty$.