Math 299Lecture 4: Chapter 2. Logic. Sections 2.4 and 2.5: Implications

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"If ..., then ..." Statements

Definition: Statements of the form "If statement A is true, then statement B is true." are called implications. Mathematically this is denoted by $A \Rightarrow B$.

$A \Rightarrow B$

- " If A then B"
- "A implies B"
- "A only if B"
- "B if A"
- " B whenever A"
- "A is sufficient for B"
- "B is necessary for A"

Examples: Determine which statement constitutes the hypothesis (assumption) and which statement is the conclusion.

1. If $x \in \mathbb{N}$, then $2x$ is even.

2. If pigs could fly, then I am on Mars.

3. The value of $x + y$ is even whenever x and y are odd.

4. I am going to carry an umbrella, only if it rains.

    – If I am going to carry an umbrella, then it means it is going to rain.

5. $x^2 < 1$ whenever $x < 1$. Note that this is a false statement!

When is the statement $A \Rightarrow B$ true?

Is the following statement true?
If pigs could fly, then I am on Mars.

"$A \Rightarrow B$" says nothing about whether A or B are true or false.

The following cases are possible implications to be true.

- A - true and B - true
- A - false and B - false
- A - false and B - true

If the assumption is false, the conclusion could be anything!

Give an example illustrating each of the above cases.

Truth table for $A \Rightarrow B$.

A	B	$A \Rightarrow B$	$\sim (A \Rightarrow B)$	$\sim B$	$A \wedge (\sim B)$
T	T				
T	F				
F	T				
F	F				

What is the negation of $A \Rightarrow B$?

Negation of if-then statement

A: "I do well in college." B: "I will get a good job."

- If A then B :

- not(If A then B) :

Theorem: The **negation** of A $\Rightarrow$ B is equivalent to A and (not B).

$$\sim(A \Rightarrow B) \equiv (A \wedge (\sim B))$$

Restate in the form of an "if-then" statement and negate the following statements.

(1) "The room is quiet, if the door is closed."

(2) "I am productive in the morning, only if I have slept well."

(3) "I am an adult, if I am 30 years old."

(4) "In order to have a driver's license, it is necessary to be at least 16 years old."

(5) "To pass MTH299, it is sufficient to have 90% on all tests and assignments."

Open Sentences

Let

$$P(x,y) : x^2 + y^2 = 4 \text{ and } Q(x,y) : \frac{y}{x} \in \mathbb{Z}$$

be open sentences with domain $A \times B$, where $A = \{1, 2\}$ and $B = \{0, \sqrt{3}\}$.

Determine for what elements in the domain the statement $P(x,y) \Rightarrow Q(x,y)$ is true.

Inverse of an if-then statement

"If I am 30 years old, then I am an adult."

The inverse of the above statement is:

"If I am not 30 years old, then I am not an adult."

Theorem: The inverse of the implication "If A, then B." is the implication "If not(A), then not(B)."

Is the inverse, in general, equivalent to the original statement?

Think of an example when a statement and its inverse are equivalent and when they are not.

Necessary Conditions

"In order to pass MTH299, it is necessary that a student completes most daily homework assignments."

What is the assumption and what is the conclusion?

Definition: A necessary condition is one that must hold in order for the result to be true. It does not guarantee that the result is true.

A is necessary for B is equivalent to B is true only if A is true, which is equivalent to $B \Rightarrow A$.

$x \in (-1, 1)$ is necessary for $x^2 - 1 < 0$.

Sufficient Conditions

"To pass MTH299, it is sufficient to have 90% on all tests and assignments."

What is the assumption and what is the conclusion?

> **Definition:** A sufficient condition is one such that if it holds, the result is guaranteed to be true. The conclusion may be true even if the condition is not satisfied.

A is sufficient for B is equivalent to $A \Rightarrow B$.

$x \in (0, 1)$ is sufficient for $x^2 - 1 < 0$.

$x \in (-1, 1)$ is sufficient for $x^2 - 1 < 0$.

Necessary and Sufficient Conditions

Fill in the blank with necessary, sufficient or necessary and sufficient.

1. $x > 1$ is ____________________ for $x^2 > 1$
2. $x \in \mathbb{N}$ is ____________________ for $x \geq 0$
3. $|x| > 1$ is ____________________ for $x^2 > 1$
4. "Mary earned an A in MTH299." is ____________________ for "Mary passed MTH299."
5. "The function f is continuous at $x = c$." is ____________________ for "The function f has a derivative at $x = c$."

Contraposition

$$A \Rightarrow B$$
"If I am 30 years old, then I am an adult."

We saw that the **inverse** of the above statement is:

$$not(A) \Rightarrow not(B)$$
"If I am not 30 years old, then I am not an adult."

and it is **not** equivalent to the original one.

Can you construct an implication using not(A) and not(B), which is equivalent to the original one?

The contrapositive of the statement $A \Rightarrow B$ is not(B)⇒not(A).

A	B	$A \Rightarrow B$	$\sim A$	$\sim B$	$(\sim B) \Rightarrow (\sim A)$
T	T				
T	F				
F	T				
F	F				

Theorem: A statement and its contrapositive are equivalent.

Find the inverse and the contrapositive of the following statements.

1. If Jane has grandchildren, then she has children.

2. If $x = 1$, then x is a solution to $x^2 - 3x + 2 = 0$.

A ⇒ B is equivalent to (not B) ⇒ (not A)

Sometimes it is easier to prove the contrapositive than it is to prove the forward statement.

Example: Prove that $\emptyset \subseteq A$, for any set A.

- If $x \in \emptyset \Rightarrow x \in A$
- Contrapositive: