Math 310-001 **Review for Final** **Fall, 2014**

I. Review homework problems.

II. Review quizzes.

III. Be able to prove short and straightforward theorems.

IV. Refer to the reviews for Exams I and II, as well as the exams themselves.

Some practice problems to review sections covered in Chapter 6 and Chapter 7

1. Show that the set of all polynomials with a constant coefficient which is divisible by 5 is an ideal in $\mathbb{Z}[x]$. On the other hand, show that the set of all polynomials with a leading coefficient which is divisible by 5 is NOT an ideal in $\mathbb{Z}[x]$. (Hint: consider $f(x) = 5x^2+2x+1$ and $g(x) = 5x^2 + x + 3$).

2. Show that the set of non-units is an ideal in $\mathbb{Z}_8$.

3. If I and J are ideals in a ring R, show that $I \cap J$ is an ideal in R. Is this the case for $I \cup J$?

4. Give an example of a subring in a given ring, which is not an ideal. Are there ideals which are not subrings?

5. If F is a field, R a nonzero ring, and $f : F \to R$ a surjective homomorphism, prove that f is an isomorphism.

6. Let $I = \{0, 5\}$ in $\mathbb{Z}_{10}$. Verify that I is an ideal. What are the elements in $\mathbb{Z}_{10}/I$? Show that $\mathbb{Z}_{10}/I \cong \mathbb{Z}_5$.

7. (a) Prove that the set T of matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a, b \in \mathbb{R}$ is a subring of $M_2(\mathbb{R})$.
   (b) Prove that the set I of matrices of the form $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ with $b \in \mathbb{R}$ is an ideal in the ring T.
   (c) What are the cosets in T/I?
   (d) Prove that $T/I \cong \mathbb{R}$.

8. Let X be a rigid rhombus in the plane, and $G = Sym(X)$ its symmetry group (consisting of rotations and reflections).
   (a) List the elements of G. Name each by a letter and sketch the symmetry it represents.
   (b) Construct the operation table for G. Is G an abelian group?
   (c) List all the subgroups H of G.

9. Let G be the set of ordered triples of integers (a, b, c) with the following operation

$$(a, b, c) * (a', b', c') = (a + a', b + b', c + c' + ab')$$

   (a) Show that G is a group under $*$.

   (b) Is G abelian?

10. Let $GL(2, \mathbb{R})$ denote the group of units in the ring $M_2(\mathbb{R})$ if 2×2 matrices with real coefficients. What is the identity element in the group $M_2(\mathbb{R})$? How about in $GL(2, \mathbb{R})$? What is the order of $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ in $GL(2, \mathbb{R})$? What is the order of $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ in $M_2(\mathbb{R})$?

11. Prove that if G is a group, its identity element is unique.

12. Let H be a subgroup of a group G. If e_G is the identity element of G and e_H is the identity element of H, prove that $e_G = e_H$.

13. Let a and n be two integers, such that $n > 1$ and $\gcd(a, n) = 1$. Let $\bar{a}$ denote the congruence class of a modulo n. Prove that $\bar{a}$ generates all of $\mathbb{Z}_n$, i.e. $\langle \bar{a} \rangle = \mathbb{Z}_n$.

14. Prove that the additive group $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not cyclic.

15. List all cyclic subgroups of (i) S_3, (ii) of U_9, (iii) of $\mathbb{Z}_9$.

16. Challenge: If $(ab)^3 = a^3b^3$ and $(ab)^5 = a^5b^5$ for all a, b in G, prove that G is abelian.