Math 320-001 **Review for the Final Exam** **Fall, 2016**

I. Be able to state important definitions and theorems.

II. Review homework problems.

III. Review quizzes and previous exams.

IV. Be able to prove short and straightforward theorems.

Important definitions you should be able to state

1. Upper bound, lower bound, supremum, infimum for a set in $\mathbb{R}$.
2. Give a rigorous definition of $\lim_{n\to\infty} s_n = L$, $\lim_{n\to\infty} s_n = +\infty$, $\lim_{n\to\infty} s_n = -\infty$.
3. Increasing, decreasing sequence.
4. Definition of limsup and liminf.
5. Cauchy sequence.
6. Continuity of a function (sequential and $\varepsilon - \delta$ property)
7. Uniform continuity
8. Limit of a function as x tends to a along a set S (Def. 20.1)
9. Two-sided limit, left hand limit, right hand limit ($\lim_{x\to a} f(x) = L$, $\lim_{x\to a^+} f(x) = L$, $\lim_{x\to a^-} f(x) = L$)
10. Define each of the following limits in terms of quantified implications ($\varepsilon - \delta$ definition, or $M - \delta$, etc.): $\lim_{x\to a^-} f(x) = +\infty$, $\lim_{x\to -\infty} f(x) = +\infty$, ...
11. Radius of convergence, interval of convergence of a power series.
12. Uniform convergence of a sequence of functions on a given set
13. Derivative of a function
14. Increasing/decreasing, strictly increasing/decreasing function

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Important theorems/axioms you should be able to state/prove

1. Proving basic properties of real numbers based on given axioms.
2. Completeness Axiom (state)
3. Proving basic properties of inf and sup (see problems 4.5-4.9, 4.14, 4.16)
4. Proving a given sequence converges using the definition (see Sect. 8)
5. Prove: Convergent sequences are bounded. (see Sect. 9)

6. Prove: If (s_n) and (t_n) converge, for $a, b \in \mathbb{R}$, prove $\lim_{n\to\infty}(as_n + bt_n) = a \lim_{n\to\infty} s_n + b \lim_{n\to\infty} t_n$. (see Sect. 9)
7. Proving a given sequence converges using limit theorems (see Sect. 9)
8. Prove: Given $s_n > 0$ for all $n \in \mathbb{N}$, prove $\lim_{n\to\infty} s_n = +\infty$ if and only if $\lim_{n\to\infty} \frac{1}{s_n} = 0$. (see Sect. 9)
9. Be able to use: All bounded monotone sequences converge.
10. Prove: If a sequence is convergent, then it is Cauchy. (Sect 10)
11. State and apply: Comparison, Ratio, Root, Alternating Series, Integral Tests
12. Prove basic properties on how to work with convergent infinite series - add, multiply by a scalar, etc. (See Problem 14.5)
13. Prove: Th. 17. 2 and 20.6 - note the differences and similarities in the proofs.
14. Prove: continuity of a function at a given point using the definition and the $\varepsilon - \delta$ properties (See Examples 1, 2, 3 in Sect. 17 and related HW.)
15. Prove: properties about operations on continuous functions (See Th. 17.4, 17.5)
16. State and apply: Extreme Value Theorem (Th. 18.1), Intermediate Value Theorem (Th. 18.2)
17. State and apply: Fundamental theorems on uniform continuity (Th. 19.2, Th. 19.4, Th. 19.5)
18. Prove: properties of limits of functions (See Th. 20.4, Th. 20.5)
19. State and apply: theorem on radius of convergence of a power series (Th. 23.1, Cor. 12.3)
20. State and apply: theorem on uniform limit of continuous functions (Th. 24.3)
21. State and apply: Weierstrass M-Test (Th. 25.5 and 25.6)
22. Prove: basic properties of derivatives (Th. 28.3(i,ii,iii))
23. State and apply: Rolle's Theorem, Mean Value Theorem
24. Prove: Cor. 29.4, 29.5, 29.7
25. State and apply: L'Hospital's Rule

In addition, these problems might help you review the material.

1. Let A be a nonempty subset of $\mathbb{R}$ that is bounded below, and $B = \{a+7 : a \in A\}$. Prove that B is also bounded below and $\inf B = 7 + \inf A$.

2. Use the definition of convergence to prove $\lim_{n\to\infty} \dfrac{3n^3+7n+1}{n^3-n-3} = 3$.

3. Let $a_n = 5 + 3(-1)^n$ for $n \in \mathbb{N}$. Prove (a_n) diverges.

4. Let $a_1 = 1$ and $a_{n+1} = \dfrac{1}{5}(3a_n + 1)$ for $n \geq 1$. Prove $\lim_{n\to\infty} a_n$ exists and find the limit.

5. Use the definition to prove that if $\lim_{n\to\infty} a_n = -\infty$ and $\lim_{n\to\infty} b_n = 7$, then $\lim_{n\to\infty}(2a_n + 3b_n) = -\infty$

6. Determine which of the following series converge. Justify your answer.
   (a) $\sum_{n=0}^{\infty} \dfrac{n^3}{5^n}$ (b) $\sum_{n=1}^{\infty} \dfrac{n-5}{2n}$ (c) $\sum_{n=1}^{\infty} \dfrac{n-5}{2n^2}$ (d) $\sum_{n=1}^{\infty} \dfrac{n+5}{2n^3}$ (e) $\sum_{n=0}^{\infty} \dfrac{3}{(-2)^n}$ - calculate the sum.
   (f) $\sum_{n=1}^{\infty} \dfrac{3(-1)^n}{n}$ (g) $\sum_{n=2}^{\infty} \dfrac{\ln(n)}{n^2}$

7. Prove that the function $\sqrt{4x+1}$ is continuous at $x_0 = 2$ by verifying the $\varepsilon - \delta$ property.

8. Prove that the function $\sqrt{4x+1}$ is uniformly continuous on $[2, \infty)$ by directly verifying the $\varepsilon - \delta$ property.

9. Prove that the function $f(x) = ((2x-1)^6 + \sqrt{3x^2+16})^5$ is uniformly continuous on $(0, 10)$ using the appropriate theorems.

10. Complete the following statements:

    (a) $\lim_{x\to a^-} f(x) = +\infty$ if and only if for any ____________ there exists ____________ such that ____________ implies ____________.

    (b) $\lim_{x\to a} f(x) = L$ if and only if for any ____________ there exists ____________ such that ____________ implies ____________.

    (c) $\lim_{x\to -\infty} f(x) = L$ if and only if for any ____________ there exists ____________ such that ____________ implies ____________.

11. Prove, using the above problem.

    (a) $\lim_{x\to 9^-} \dfrac{7}{9-x} = +\infty$.

    (b) $\lim_{x\to 10} \dfrac{7}{9-x} = -7$. (How could we prove this in a simpler way than using the above problem?)

    (c) $\lim_{x\to \infty} \dfrac{7x}{9-x} = -7$

12. For each of the following series find the radius of convergence and determine the exact interval of convergence.
    (a) $\sum_{n=1}^{\infty} \dfrac{2^n}{n^2} x^n$ (b) $\sum_{n=1}^{\infty} \dfrac{5^n}{n!} x^n$ (c) $\sum_{n=1}^{\infty} \dfrac{7^n}{n5^n} x^n$ (d) $\sum_{n=1}^{\infty} 5^{-n} x^{2n}$

13. For $x \in \mathbb{R}$, let $f_n(x) = \dfrac{x}{n^2+1}$.

   (a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

   (b) Determine whether $f_n \to f$ uniformly on $[-5, 10]$.

   (c) Determine whether $f_n \to f$ uniformly on $[5, \infty)$.

14. For $x \in [0, \infty)$, let $f_n(x) = \dfrac{1}{x^n+1}$.

   (a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

   (b) Determine whether $f_n \to f$ uniformly on $[0, 5]$.

15. Does $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ represent a continuous function on $[-10\,10]$? Explain your reasoning.

16. Use the definition of derivative to calculate

   (a) $f'(1)$ for $f(x) = \sqrt{3x+6}$.

   (b) $g'(0)$ for $g(x) = x^2 \sin(\frac{1}{x})$ if $x \neq 0$ and $g(0) = 0$.

17. Use the Mean Value Theorem to show $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

18. Show $ex \leq e^x$ for all $x \in \mathbb{R}$.

19. Find $\lim_{x\to 0}(1+5x)^{1/x}$.

20. Find $\lim_{x\to 0}(1+5x)^{x/(x+1)}$.

21. Find $\lim_{x\to\infty}(e^x + x)^{1/x}$.