Math 320-002 **Review for Exam 2** **Spring, 2017**

This exam focuses on Sections 11, 12, 14, 15, 17, 18, 19, 20.

I. Be able to state important definitions and theorems.

II. Review homework problems and in-class worksheets.

III. Review quizzes.

IV. Be able to prove short and straightforward theorems.

Important definitions you should be able to state

1. Subsequences and their properties.
2. Convergence of an infinite series
3. Continuity of a function (sequential and $\varepsilon - \delta$ property)
4. Uniform continuity
5. Limit of a function as x tends to a along a set S (Def. 20.1)
6. Two-sided limit, left hand limit, right hand limit ($\lim_{x\to a} f(x) = L$, $\lim_{x\to a^+} f(x) = L$, $\lim_{x\to a^-} f(x) = L$)
7. Define each of the following limits in terms of quantified implications ($\varepsilon - \delta$ definition, or $M - \delta$, etc.): $\lim_{x\to a^-} f(x) = +\infty$, $\lim_{x\to -\infty} f(x) = +\infty$, ...

Important theorems you should be able to state/prove

1. State and apply: Bolzano-Weierstrass Theorem
2. State and apply: Comparison, Ratio, Root, Alternating Series, Integral Tests
3. Prove basic properties on how to work with convergent infinite series - add, multiply by a scalar, etc. (See Problem 14.5)
4. Prove: Th. 17. 2 and 20.6 - note the differences and similarities in the proofs.
5. Prove: continuity of a function at a given point using the definition and the $\varepsilon - \delta$ properties (See Examples 1, 2, 3 in Sect. 17 and related HW.)
6. Prove: properties about operations on continuous functions (See Th. 17.4, 17.5)
7. State and apply: Extreme Value Theorem (Th. 18.1), Intermediate Value Theorem (Th. 18.2)
8. State and apply: Fundamental theorems on uniform continuity (Th. 19.2, Th. 19.4, Th. 19.5)
9. Prove: properties of limits of functions (See Th. 20.4, Th. 20.5)

In addition, these problems might help you review the material.

1. Determine which of the following series converge. Justify your answer.
   (a) $\sum_{n=0}^{\infty} \frac{n^3}{5^n}$ (b) $\sum_{n=1}^{\infty} \frac{n-5}{2n}$ (c) $\sum_{n=1}^{\infty} \frac{n-5}{2n^2}$ (d) $\sum_{n=1}^{\infty} \frac{n+5}{2n^3}$ (e) $\sum_{n=0}^{\infty} \frac{3}{(-2)^n}$ - calculate the sum.
   (f) $\sum_{n=1}^{\infty} \frac{3(-1)^n}{n}$ (g) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$.

2. Prove that the function f defined by $f(x) = 3$, if $x \in \mathbb{Q}$ and $f(x) = 5$, if $x \in \mathbb{R} \setminus \mathbb{Q}$ is discontinuous at every point $x \in \mathbb{R}$.

3. Prove that the function
$$f(x) = \begin{cases} x^2 - 3, & x > 3 \\ x + 5, & x \leq 3. \end{cases}$$
   is discontinuous at $x = 3$.

4. Prove that the function
$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} + 7, & x \neq 0 \\ 7, & x = 0. \end{cases}$$
   is continuous at $x = 0$.

5. Use the sequential definition of continuity to show $f(x) = 3x^2 - 5$ is continuous on $\mathbb{R}$.

6. Use the $\varepsilon - \delta$ property to show $f(x) = 3x^2 - 5$ is continuous on $\mathbb{R}$.

7. Show that the function f defined by $f(x) = \cos(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$ is discontinuous $x = 0$.

8. Prove that the function $\sqrt{4x+1}$ is continuous at $x_0 = 2$ by verifying the $\varepsilon - \delta$ property.

9. Prove that the function $\sqrt{4x+1}$ is uniformly continuous on $[2, \infty)$ by directly verifying the $\varepsilon - \delta$ property.

10. Prove that the function $f(x) = ((2x-1)^6 + \sqrt{3x^2+16})^5$ is uniformly continuous on $(0, 10)$ using the appropriate theorems.

11. Prove that the function $f(x) = \frac{5}{x-3}$ is NOT uniformly continuous on $(3, 4)$, using the appropriate theorems.

12. Complete the following statements:
    (a) $\lim_{x \to a^-} f(x) = +\infty$ if and only if for any ______________ there exits ______________ such that ______________ implies ______________.
    (b) $\lim_{x \to a} f(x) = L$ if and only if for any ______________ there exits ______________ such that ______________ implies ______________.
    (c) $\lim_{x \to -\infty} f(x) = L$ if and only if for any ______________ there exits ______________ such that ______________ implies ______________.

13. Prove, using the above problem.
    (a) $\lim_{x \to 9^-} \frac{7}{9-x} = +\infty$.

(b) $\lim_{x \to 10} \frac{7}{9-x} = -7$. (How could we prove this in a simpler way than using the above problem?)

(c) $\lim_{x \to \infty} \frac{7x}{9-x} = -7$