

- I. Be able to state important definitions and theorems.
  - II. Review homework problems.
  - III. Review quizzes and previous exams.
  - IV. Be able to prove short and straightforward theorems.
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### Important definitions you should be able to state

1. Upper bound, lower bound, supremum, infimum for a set in  $\mathbb{R}$ .
  2. Give a rigorous definition of  $\lim_{n \rightarrow \infty} s_n = L$ ,  $\lim_{n \rightarrow \infty} s_n = +\infty$ ,  $\lim_{n \rightarrow \infty} s_n = -\infty$ .
  3. Increasing, decreasing sequence.
  4. Definition of limsup and liminf.
  5. Cauchy sequence.
  6. Continuity of a function (sequential and  $\varepsilon - \delta$  property)
  7. Uniform continuity
  8. Limit of a function as  $x$  tends to  $a$  along a set  $S$  (Def. 20.1)
  9. Two-sided limit, left hand limit, right hand limit ( $\lim_{x \rightarrow a} f(x) = L$ ,  $\lim_{x \rightarrow a^+} f(x) = L$ ,  $\lim_{x \rightarrow a^-} f(x) = L$ )
  10. Define each of the following limits in terms of quantified implications ( $\varepsilon - \delta$  definition, or  $M - \delta$ , etc.):  $\lim_{x \rightarrow a^-} f(x) = +\infty$ ,  $\lim_{x \rightarrow -\infty} f(x) = +\infty$ , ...
  11. Radius of convergence, interval of convergence of a power series.
  12. Uniform convergence of a sequence of functions on a given set
  13. Derivative of a function
  14. Increasing/decreasing, strictly increasing/decreasing function
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### Important theorems/axioms you should be able to state/prove

1. Proving basic properties of real numbers based on given axioms.
2. Completeness Axiom (state)
3. Proving basic properties of inf and sup (see problems 4.5-4.9, 4.14, 4.16)
4. Proving a given sequence converges using the definition (see Sect. 8)
5. Prove: Convergent sequences are bounded. (see Sect. 9)

6. Prove: If  $(s_n)$  and  $(t_n)$  converge, for  $a, b \in \mathbb{R}$ , prove  $\lim_{n \rightarrow \infty} (as_n + bt_n) = a \lim_{n \rightarrow \infty} s_n + b \lim_{n \rightarrow \infty} t_n$ . (see Sect. 9)
7. Proving a given sequence converges using limit theorems (see Sect. 9)
8. Prove: Given  $s_n > 0$  for all  $n \in \mathbb{N}$ , prove  $\lim_{n \rightarrow \infty} s_n = +\infty$  if and only if  $\lim_{n \rightarrow \infty} \frac{1}{s_n} = 0$ . (see Sect. 9)
9. Be able to use: All bounded monotone sequences converge.
10. Prove: If a sequence is convergent, then it is Cauchy. (Sect 10)
11. State and apply: Comparison, Ratio, Root, Alternating Series, Integral Tests
12. Prove basic properties on how to work with convergent infinite series - add, multiply by a scalar, etc. ( See Problem 14.5)
13. Prove: Th. 17. 2 and 20.6 - note the differences and similarities in the proofs.
14. Prove: continuity of a function at a given point using the definition and the  $\varepsilon - \delta$  properties (See Examples 1, 2, 3 in Sect. 17 and related HW.)
15. Prove: properties about operations on continuous functions (See Th. 17.4, 17.5)
16. State and apply: Extreme Value Theorem (Th. 18.1), Intermediate Value Theorem (Th. 18.2)
17. State and apply: Fundamental theorems on uniform continuity (Th. 19.2, Th. 19.4, Th. 19.5)
18. Prove: properties of limits of functions (See Th. 20.4, Th. 20.5)
19. State and apply: theorem on radius of convergence of a power series (Th. 23.1, Cor. 12.3)
20. State and apply: theorem on uniform limit of continuous functions (Th. 24.3)
21. State and apply: Weierstrass M-Test (Th. 25.5 and 25.6)
22. Prove: basic properties of derivatives and differentiable functions (Th. 28.2, Th. 28.3(i,ii,iii, iv))
23. State and apply: Rolle's Theorem, Mean Value Theorem
24. Prove: Cor. 29.4, 29.5, 29.7
25. State and apply: L'Hospital's Rule

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**In addition, these problems might help you review the material.**

- Let  $A$  be a nonempty subset of  $\mathbb{R}$  that is bounded below, and  $B = \{a + 7 : a \in A\}$ . Prove that  $B$  is also bounded below and  $\inf B = 7 + \inf A$ .
- Let  $A$  be a nonempty subset of  $\mathbb{R}$  that is bounded below, and  $B = \{-2a : a \in A\}$ . Prove that  $B$  is bounded above and  $\sup B = -2 \inf A$ .
- Use the definition of convergence to prove  $\lim_{n \rightarrow \infty} \frac{3n^3 + 7n + 1}{n^3 - n - 3} = 3$ .
- Let  $a_n = 5 + 3(-1)^n$  for  $n \in \mathbb{N}$ . Prove  $(a_n)$  diverges.
- Let  $a_1 = 1$  and  $a_{n+1} = \frac{1}{5}(3a_n + 1)$  for  $n \geq 1$ . Prove  $\lim_{n \rightarrow \infty} a_n$  exists and find the limit.
- Use the definition to prove that if  $\lim_{n \rightarrow \infty} a_n = -\infty$  and  $\lim_{n \rightarrow \infty} b_n = 7$ , then  $\lim_{n \rightarrow \infty} (2a_n + 3b_n) = -\infty$ .
- Determine which of the following series converge. Justify your answer.  
(a)  $\sum_{n=0}^{\infty} \frac{n^3}{5^n}$  (b)  $\sum_{n=1}^{\infty} \frac{n-5}{2n}$  (c)  $\sum_{n=1}^{\infty} \frac{n-5}{2n^2}$  (d)  $\sum_{n=1}^{\infty} \frac{n+5}{2n^3}$  (e)  $\sum_{n=0}^{\infty} \frac{3}{(-2)^n}$  - calculate the sum.  
(f)  $\sum_{n=1}^{\infty} \frac{3(-1)^n}{n}$  (g)  $\sum_{n=1}^{\infty} (5 + \frac{1}{n})2^{-n}$
- Prove that the function  $\sqrt{4x+1}$  is continuous at  $x_0 = 2$  by verifying the  $\varepsilon - \delta$  property.
- Prove that the function  $\sqrt{4x+1}$  is uniformly continuous on  $[2, \infty)$  by directly verifying the  $\varepsilon - \delta$  property.
- Prove that the function  $f(x) = ((2x-1)^6 + \sqrt{3x^2+16})^5$  is uniformly continuous on  $(0, 10)$  using the appropriate theorems.
- Complete the following statements:  
(a)  $\lim_{x \rightarrow a^-} f(x) = +\infty$  if and only if for any \_\_\_\_\_ there exists \_\_\_\_\_ such that \_\_\_\_\_ implies \_\_\_\_\_.  
(b)  $\lim_{x \rightarrow a} f(x) = L$  if and only if for any \_\_\_\_\_ there exists \_\_\_\_\_ such that \_\_\_\_\_ implies \_\_\_\_\_.  
(c)  $\lim_{x \rightarrow -\infty} f(x) = L$  if and only if for any \_\_\_\_\_ there exists \_\_\_\_\_ such that \_\_\_\_\_ implies \_\_\_\_\_.
- Prove, using the above problem.  
(a)  $\lim_{x \rightarrow 9^-} \frac{7}{9-x} = +\infty$ .  
(b)  $\lim_{x \rightarrow 10} \frac{7}{9-x} = -7$ . (How could we prove this in a simpler way than using the above problem?)  
(c)  $\lim_{x \rightarrow \infty} \frac{7x}{9-x} = -7$
- For each of the following series find the radius of convergence and determine the exact interval of convergence.  
(a)  $\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$  (b)  $\sum_{n=1}^{\infty} \frac{5^n}{n!} x^n$  (c)  $\sum_{n=1}^{\infty} \frac{7^n}{n5^n} x^n$  (d)  $\sum_{n=1}^{\infty} 5^{-n} x^{2n}$

14. For  $x \in \mathbb{R}$ , let  $f_n(x) = \frac{x}{n^2 + 1}$ .
- (a) Find  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ .
  - (b) Determine whether  $f_n \rightarrow f$  uniformly on  $[-5, 10]$ .
  - (c) Determine whether  $f_n \rightarrow f$  uniformly on  $[5, \infty)$ .
15. For  $x \in [0, \infty)$ , let  $f_n(x) = \frac{1}{x^n + 1}$ .
- (a) Find  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ .
  - (b) Determine whether  $f_n \rightarrow f$  uniformly on  $[0, 5]$ .
16. Does  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  represent a continuous function on  $[-10, 10]$ ? Explain your reasoning.
17. Use the definition of derivative to calculate
- (a)  $f'(1)$  for  $f(x) = \sqrt{3x + 6}$ .
  - (b)  $g'(0)$  for  $g(x) = x^2 \sin(\frac{1}{x})$  if  $x \neq 0$  and  $g(0) = 0$ .
18. Use the Mean Value Theorem to show  $|\sin x - \sin y| \leq |x - y|$  for all  $x, y \in \mathbb{R}$ .
19. Show  $ex \leq e^x$  for all  $x \in \mathbb{R}$ .
20. Find  $\lim_{x \rightarrow 0} (1 + 5x)^{1/x}$ .
21. Find  $\lim_{x \rightarrow 0} (1 + 5x)^{x/(x+1)}$ .
22. Find  $\lim_{x \rightarrow \infty} (e^x + x)^{1/x}$ .