The View from Here

Confronting Analysis

Tina Rapke

When I think of my experience learning analysis, there is a mixture of emotions. There are great feelings of accomplishment and perseverance that are seeded in feelings of anxiousness, discouragement, and inadequacy. Before analysis, I found math enjoyable and interesting. Never before had it left a bad taste in my mouth. I took math courses because I found them easy and they were good GPA boosters. Analysis would change all of that. It was unlike any other math I had ever seen—to be honest, I wasn't even sure if it was math.

As I walked into the first lecture of my first analysis class, I was excited but had those first-day butterflies: Will the course be interesting? Will I catch on quickly? Will the professor be impressed with my abilities? I had taken a course from this professor before, so that was not causing me too much worry, but the text, *Principles of Mathematical Analysis*, by Walter Rudin, didn't look like anything I had seen before.

Then the professor walked in and began to talk about Dedekind cuts and other obscure-sounding mathematical notions—density, orderings, countability. It didn't take long for me to realize I was in way over my head. I had jumped into the deep end, and I was drowning quickly. Before this class, I never understood my friends' reactions when I told them that I was majoring in mathematics: "I'm not a math person," "It's too hard," "I have math dyslexia." Now here I was, reconsidering my own choice of major, about to give up on my hopes of becoming a mathematician. It was as if I was taking the math ride; it was fun, it was good, but now I had to get off the bus. I had reached the end of my abilities.

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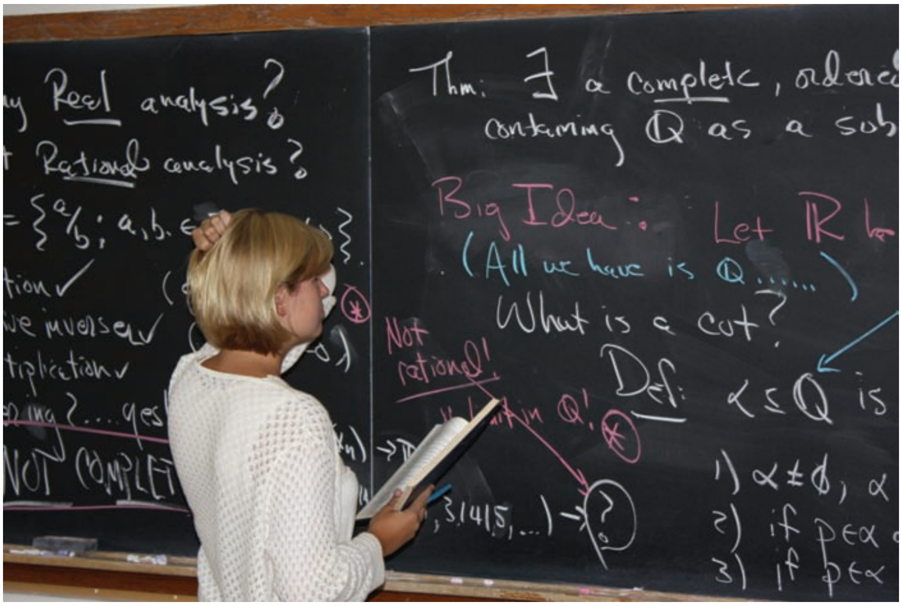
Not being the type of person to give up easily, and with graduation so close, I knew I had no choice but to complete the course. And I did—with hard work and extraordinary effort. I've since gone on to graduate school and even passed a candidacy examine in analysis! I am sharing my story because I believe that my initial reaction to analysis is not unique, and the strategies I used to make it through this experience could be helpful to others.



Irrational Thoughts

My struggles began with the very first assignment. Looking back at my notes from that term, I see that we were asked to show that the set of positive rationals with squares greater than 2 has no least element. I stumbled through the proof. It was clear that I was trying to mimic a similar proof from the class notes. Did I really understand this argument? Apparently not! The difficulties I was having are glaringly obvious when I look at the second assignment. There we were asked to do the following problem from page 44 of Rudin:

"Regard Q, the set of all rational numbers, as a metric space, with $d(p, q) = | p - q |$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q but that E is not compact."

The first line in my response was, "Let s be the least rational such that $s^2 > 2$."

Wow, that's amazing because I had just proved that no such least element exists! This demonstrates how confused I really was during the course—proving something one day and then claiming the opposite on the very next assignment.

I remember a conversation with the professor after receiving a poor mark on the assignment. We talked about how I thought I could arrange the rationals in order. Part of my confusion came with the introduction of the concept of "countability." I knew that a set was countable if there was a one-to-one correspondence between the set and the natural numbers, and we had learned that the rationals were a countable set, which meant that they could be arranged in a sequence: $r_1, r_2, r_3, \ldots$. Now, given any collection of natural numbers, it is always possible to go through and pick out the smallest one, so it seemed to me that given a rational number r_n on my list, I could go through the list of remaining rationals and let r_{n+1} be the "next largest one." At the professor's request, I experimented with trying to name the "next largest" rational, and as you can guess, I wasn't able to do so.

I remember thinking in terms of "thickness," which turned out to be a temporary stand-in for density. Given a rational, I could not name the next one. I related this to the same property of the real numbers, so the rational numbers and the real numbers were similar in terms of "thickness." But then this blurred into my understanding of cardinality and suggested to me that the rationals had a cardinality close to that of the reals. But the real numbers are uncountable and the rationals are countable. Why?

Looking back at all the course assignments, I remember feeling tense and anxious. My stomach was constantly tied in knots. I wanted to run away and pretend the assignments never existed. I felt uneasy because I knew I wouldn't be able to complete them. I didn't understand what was really going on. If I didn't grasp the concepts, how could I complete the assignments?

Making Progress

Rudin's definition says a set is countable if there is a one-to-one correspondence with the natural numbers, which for me meant that to show A is countable I had to find an explicit function from the naturals to the set A. But Rudin explains that the rationals are countable in a less direct manner. He first provides the following result: Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ where each $a_k \in A$. Then B_n is countable. After establishing this, he notes that we can apply this theorem with $n = 2$ by observing that the rational numbers are of the form a/b where a and b are integers.

This is all well and good, but when first learning about countability, I was confused because I wanted to see a correspondence with the natural numbers via some *explicit* function. Many years later, I found the following theorem in *Principles in Real Analysis* by Aliprantis: For an infinite set A the following statements are equivalent:

(i) A is countable; (ii) There exists a subset $B \subseteq N$ (where N is the set of natural numbers) and a function $f: B \to A$ that is onto; (iii) There exists a function $g: A \to N$ that is one-to-one.

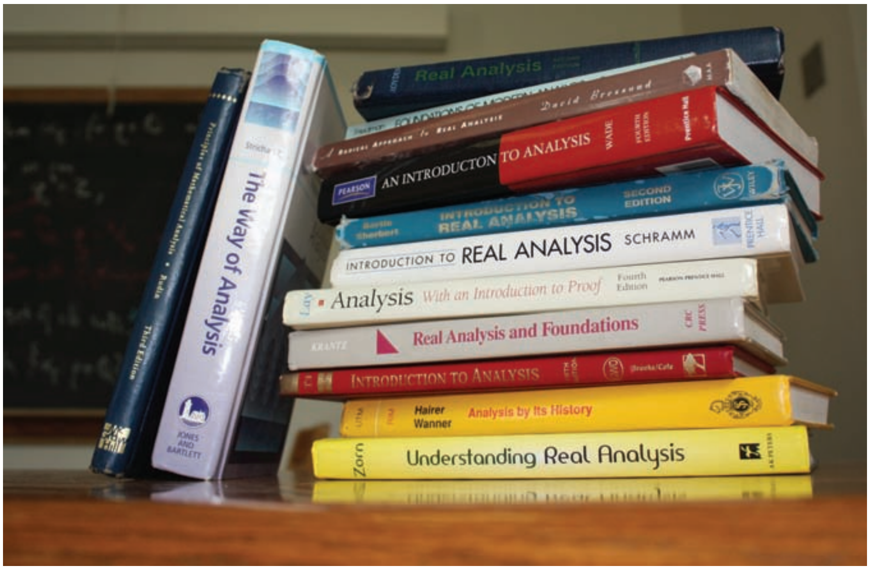
Currently, I like justifying that the rationals are countable by using the following steps:

1. First consider the set P of positive rationals.
2. Let $B = \{2^n 3^m : n, m \in N\}$ which is a subset of N.
3. Construct the function $f: B \to P$ by letting $f(2^n 3^m) = n/m$.
4. Appeal to the above theorem.
5. Finally, bring in the negative rationals by noting that the union of two countable sets is countable.

My Story Now

I completed my first analysis course, did fairly well, and received a scholarship as a consequence. I've written my Ph.D. candidacy exam in analysis and passed. One of my favorite memories of studying was one night when analysis crept into my dreams. I woke up in a panicky cold sweat. In my dream I was being chased by some analysis monster. My only defense was to use the "blancmange function" (a continuous but nowhere differentiable function discussed in Hairer and Wanner's book *Analysis by Its History*) as a boomerang. I took it as a good sign at the time that analysis concepts were finding their way into my subconscious.

The blancmange function



Photographs courtesy of Stephen Abbott

While studying for my candidacy exam, I used several different textbooks. I can't say with certainty that my initial misconceptions with density and countability would have been resolved by seeing the theorems from Aliprantis's *Principles in Real Analysis* sooner, but this book fundamentally changed my understanding of these ideas—which brings up an important point about textbooks: Before encountering analysis, I never consulted different sources. If you're having trouble in math, you should consider finding alternate textbooks and other resources. I like Aliprantis and Burkinshaw because of the way that they lay things out and because they have a very conversational style. They use the word "we" a lot and make it seem that the proofs are a team effort. *Understanding Analysis* by Stephen Abbott is another good resource. It is also very well laid out, and the author doesn't give things up; he makes you work, but he guides you and gives many strong hints. I really enjoy the discussions at the beginning of each section, which include some history and motivation. I also like *Analysis by Its History* because it provides some well-known counterexamples, and I like learning through history. As for a first text in analysis, I recommend *A Friendly Introduction to Analysis* by Witold Kosmala because it has many figures and examples. However, over all of these, I find Rudin to be the most useful reference text. Rudin goes straight to the point with very few fillers. I like to think that Rudin provides a good struggle—you need to read between the lines, fill in the gaps, and try really hard.

Since that first course, I've gone on to take many other analysis courses, including measure theory and functional analysis, and I've done quite well. It took hard work, but I've come to appreciate the beauty and proof techniques of analysis, and I now find it fascinating.

In the end, if you're having trouble with analysis and find yourself in this story, YOU ARE NOT ALONE! I've heard many other successful graduate students say that they had similar issues when they first confronted analysis. So keep with it. Some struggles reap big rewards!

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