Math 421-001 **Exam 2** **April 10, 2015**

Name: ______________________ PID: A______________

1. *(3 points each)* Define the following concepts:

   (a) **Interior** of a set $E \subseteq \mathbb{R}^n$.

   $E^o = \bigcup \{ U \subseteq \mathbb{R}^n : U\text{-open and } U \subseteq E \}$

   (b) **Closure** of a set $E \subseteq \mathbb{R}^n$.

   $\overline{E} = \bigcap \{ V \subseteq \mathbb{R}^n : V\text{-closed and } V \supseteq E \}$

   (c) **Boundary** of a set $E \subseteq \mathbb{R}^n$.

   $\partial E = \{ \vec{x} \in \mathbb{R}^n : \forall r > 0 \ B_r(\vec{x}) \cap E \neq \phi$ and $B_r(\vec{x}) \cap E^c \neq \phi \}$.

   (d) The set $E \subseteq \mathbb{R}^n$ is **compact**, iff for every open cover of E, there exists a finite subcover, i.e., if U_α -open $\forall \alpha \in A$ and $\bigcup_{\alpha \in A} U_\alpha \supseteq E$, then $\exists \{\alpha_1, \dots, \alpha_n\} \subseteq A : \bigcup_{i=1}^{n} U_{\alpha_i} \supseteq E$.

   (e) The sets U and V **separate** the set $E \subseteq \mathbb{R}^n$.

   $U, V \neq \phi$, $U, V \subseteq E$, U, V - rel open in E, $U \cup V = E$, and $U \cap V = \phi$.

   (f) The set $E \subseteq \mathbb{R}^n$ is **connected**.

   E is connected iff there exist no sets U, V, which separate it.

   (g) Let $E \subseteq \mathbb{R}^n$. The function $\mathbf{f} : E \to \mathbb{R}^m$ is **continuous** at a point $\mathbf{a} \in E$, iff

   $\forall \varepsilon > 0 \ \exists \delta > 0 : \ \|\vec{x} - \vec{a}\| < \delta$ and $\vec{x} \in E \Rightarrow \|f(\vec{x}) - f(\vec{a})\| < \varepsilon$.

   (h) Let $E \subseteq \mathbb{R}^n$. The function $\mathbf{f} : E \to \mathbb{R}^m$ is **uniformly continuous** on E, iff.

   $\forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall \vec{x}, \vec{y} \in E \ \|\vec{x} - \vec{y}\| < \delta \Rightarrow \|f(\vec{x}) - f(\vec{y})\| < \varepsilon$.

2. *(15 points)* Suppose that $A \subseteq B \subseteq \mathbb{R}^n$. Prove that the interior of A is a subset of the interior of B, i.e. $A^\circ \subseteq B^\circ$.

Assume $A \subseteq B$. Recall that for any set A, $A^\circ \subseteq A$. Thus $A^\circ \subseteq B$ and A° is an open set. Thus $A^\circ \subseteq B^\circ$, as B° is the union of all open sets contained in B.

$\blacksquare$

3. *(15 points)* Suppose C is a compact set in $\mathbb{R}^n$ and assume that for every $\mathbf{x} \in C$, there exists $\delta_{\mathbf{x}} > 0$ such that $B_{\delta_{\mathbf{x}}}(\mathbf{x}) \cap C = \{\mathbf{x}\}$. Prove that C is a finite set.

Note that $\forall\, \underline{x} \in C$, $B_{\delta_x}(\underline{x})$ is an open set and

$\bigcup_{\underline{x} \in C} B_{\delta_x}(\underline{x}) \supseteq C$. Thus $\{B_{\delta_x}(\underline{x})\}_{\underline{x} \in C}$ is an open cover of C. Since C is compact, every open cover has a finite subcover, thus $\exists\ \underline{x}_1, \ldots, \underline{x}_n \in C$ s.t. $C \subseteq \bigcup_{i=1}^{n} B_{\delta_{x_i}}(\underline{x}_i)$.

Therefore, $C \subseteq \bigcup_{i=1}^{n} \left(B_{\delta_{x_i}}(\underline{x}_i) \cap C\right) = \{\underline{x}_1, \ldots, x_n\}$.

Thus $C \subseteq \{x_1, \ldots, x_n\} \subseteq C$, so $C = \{\underline{x}_1, \ldots, \underline{x}_n\}$,

and so $|C| = n$, i.e. C is finite.

4. *(15 points)* Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x,y,z) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2+z^2}, & \text{if } (x,y,z) \neq (0,0,0) \\ 0, & \text{if } (x,y,z) = (0,0,0). \end{cases}$$

Prove that f is continuous on $\mathbb{R}^3$.

Let $\varepsilon > 0$ be given, arbitrary.

Let $\delta = \varepsilon/2$ and assume $0 < \|(x,y,z) - (0,0,0)\| < \delta$,

i.e. $0 < \sqrt{x^2+y^2+z^2} < \delta$.

$$|f(x,y,z) - f(0,0,0)| = \left|\frac{x^3-y^3}{x^2+y^2+z^2}\right| \leq |x| \cdot \frac{x^2}{x^2+y^2+z^2} + |y| \cdot \frac{y^2}{x^2+y^2+z^2}$$

$$\leq |x| + |y| \leq \sqrt{x^2+y^2+z^2} + \sqrt{x^2+y^2+z^2} < 2\delta = \varepsilon.$$

Thus $\forall \varepsilon > 0$ $\exists$ $\delta > 0$ s.t. if $\|(x,y,z) - (0,0,0)\| < \delta$, then $|f(x,y,z) - f(0,0,0)| < \varepsilon$, i.e. f is continuous at $(0,0,0)$.

Note that $\forall$ $(x,y,z) \neq (0,0,0)$ f is continuous at this point, since it is given by the ratio of 2 polynomials and the denominator is nonzero.

Thus, f is continuous on $\mathbb{R}^3$.

5. *(15 points)* Let $n \in \mathbb{N}$ and assume $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ are convergent sequences in $\mathbb{R}^n$. Prove that

$$\lim_{k\to\infty}(\mathbf{x}_k \cdot \mathbf{y}_k) = (\lim_{k\to\infty}\mathbf{x}_k)\cdot(\lim_{k\to\infty}\mathbf{y}_k).$$

Let $\underline{x}_k \xrightarrow[k\to\infty]{} \underline{x}$ and $y_k \xrightarrow[k\to\infty]{} y$. We claim $\underline{x}_k \cdot y_k \xrightarrow[k\to\infty]{} \underline{x}\cdot y$. Indeed,

let $\varepsilon > 0$ be given. Since $\{y_k\}$ converges, it is bounded, so $\exists\, M \in \mathbb{R}, M > 0$ s.t. $\|y_k\| < M$ $\forall k$.

Assume $\underline{x} \neq \underline{0}$.

Since $\underline{x}_k \xrightarrow[k\to\infty]{} \underline{x}$ $\exists\, N_1 : \forall k > N_1 : \|\underline{x}_k - \underline{x}\| < \dfrac{\varepsilon}{2M}$

and since $y_k \xrightarrow[k\to\infty]{} y$ $\exists\, N_2 : \forall k > N_2 : \|y_k - y\| < \dfrac{\varepsilon}{2\|x\|}$

Let $N = \max\{N_1, N_2\}$, assume $k > N$.

$$|\underline{x}_k \cdot y_k - \underline{x}\cdot y| = |\underline{x}_k \cdot y_k - \underline{x}\cdot y_k + \underline{x}\cdot y_k - \underline{x}\cdot y|$$

Triangle in.

$$\leq |\underline{x}_k \cdot y_k - \underline{x}\cdot y_k| + |\underline{x}\cdot y_k - \underline{x}\cdot y|$$

Cauchy-Schwartz

$$\leq \|\underline{x}_k - \underline{x}\|\,\|y_k\| + \|\underline{x}\|\,\|y_k - y\|$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Now, if $\underline{x} = \underline{0}$, as above we arrive at

$$|\underline{x}_k \cdot y_k - 0| \leq \|\underline{x}_k - \underline{x}\|\cdot\|y_k\| < \varepsilon$$

as long as $k : \|\underline{x}_k - \underline{x}\| < \dfrac{\varepsilon}{M}$.

6. *(16 points)* Let $E \subset \mathbb{R}^n$. Assume $\mathbf{x} \in \mathbb{R}^n$ is such that for all $r > 0$, $B_r(\mathbf{x}) \cap E^c \neq \emptyset$. Prove that $\mathbf{x} \notin E^\circ$.

Assume, by way of contradiction $\vec{x} \in E^\circ$.

Note that E° - open, so $\exists\ r^* > 0$ s.t

$B_{r^*}(\vec{x}) \subseteq E^\circ \subseteq E$.

Thus $B_{r^*}(\vec{x}) \cap E^c = \phi$, which contradicts the assumption, so $\vec{x} \notin E^\circ$.

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Do not write in the area below. (For recording YOUR SCORES only.)

1. ________	OUT OF 24	4. ________	OUT OF 15
2. ________	OUT OF 15	5. ________	OUT OF 15
3. ________	OUT OF 15	Bonus ________	OUT OF 16

TOTAL: ________ OUT OF 100