Math 421-001 **Homework Assignment #5** **Due Feb. 11, 2015**

Name ______________________________

Definition 1. A function $||\cdot||$ from $\mathbb{R}^n$ to $\mathbb{R}$ is called a **norm**, provided the following three conditions are satisfied.

(i) For all $\mathbf{x} \in \mathbb{R}^n$, $||\mathbf{x}|| \geq 0$ with equality only when $\mathbf{x} = \mathbf{0}$.

(ii) For any $\alpha \in \mathbb{R}$ and for any $\mathbf{x} \in \mathbb{R}^n$, $||\alpha\mathbf{x}|| = |\alpha|||\mathbf{x}||$.

(iii) (**Triangle Inequality.**) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

Definition 2. Two norms p and q on a vector space V are **equivalent** if there exist two real constants c and C, with $c > 0$ such that for every vector $\mathbf{v}$ in V, the following inequalities hold:

$$cq(\mathbf{v}) \leq p(\mathbf{v}) \leq Cq(\mathbf{v}).$$

Problem 1. Consider the following functions from $\mathbb{R}^n$ to $\mathbb{R}$:

(a) The ℓ^1-norm, $||\cdot||_1 : \mathbb{R}^n \to \mathbb{R}$, defined by

$$||\mathbf{x}||_1 := \sum_{i=1}^{n} |x_i|.$$

(b) The ℓ^∞-norm, $||\cdot||_\infty : \mathbb{R}^n \to \mathbb{R}$, defined by

$$||\mathbf{x}||_\infty := \max_{1 \leq i \leq n} \{|x_i|\}.$$

Prove that they are both norms on $\mathbb{R}^n$ by showing that they satisfy the three conditions in the definition of a norm. You can use the fact that the Triangle Inequality holds on $\mathbb{R}$, i.e. if $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

Problem 2. Show that $||\cdot||_\infty$ and $||\cdot||_1$ satisfy the following inequality on $\mathbb{R}^n$:

$$||\mathbf{x}||_\infty \leq ||\mathbf{x}||_1 \leq n||\mathbf{x}||_\infty,$$

and conclude that these two norms are equivalent.