RESEARCH PROPOSAL

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1. Summary

My main research interests lie in low dimensional topology and geometry, knot theory, and handlebody theory. More specifically, these interests include Heegaard Floer homology, sutured Floer homology, and Heegaard splittings and tunnel number of knots.

Heegaard Floer theory is a set of invariants of three- and four-dimensional manifolds which have significantly impacted the study of many areas of low dimensional topology including Dehn surgery and foliation theory. As can be seen in Sections 3 and 4, my current and future work aims to fit within this framework.

One important aspect of the theory is that it provides invariants for knots in three-manifolds [25, 28]. To date, my research has focused primarily on the interactions between knot theory and various Floer homology invariants.

Inspired by the theory of sutured manifolds [5], Juhász introduced sutured Floer homology, an invariant of three-manifolds with certain types of boundary [13]. In [32] I use this invariant to distinguish minimal genus Seifert surfaces of an infinite family of knots with trivial Alexander polynomial. This family of examples provides the first use of the full strength of sutured Floer homology, and not merely its Euler characteristic (a classical torsion), to distinguish Seifert surfaces.

The three-manifolds with simplest Heegaard Floer invariants are rational homology three-spheres for which the rank of the Floer homology is equal to the order of the first singular homology. Stemming from the fact that lens spaces form a large collection of examples, such manifolds are called *L-spaces*. It has been a long standing goal in three-manifold topology to classify the knots on which surgery can be performed to yield a lens space. Indeed, there is a conjecture that a construction due to Berge which produces knots in S^3 with lens space surgeries is complete (in the sense that any knot admitting a lens space surgery comes from this construction). Then it is natural to look beyond Berge's list for *L-space knots*, i.e., knots that admit L-space surgeries [17, 27, 34]. In [33] I find a family of L-space knots, some of which are known to live outside of Berge's collection. In particular, I classify the knots in a subfamily of twisted torus knots that admit L-space surgeries.

A step in a similar direction is my result with Hom and Lidman [12]. In this work, we find satellite operations on knots, using *Berge-Gabai knots* as the pattern, that produce L-space knots. From their definition, Berge-Gabai knots are knots in $S^1 \times D^2$ with non-trivial solid tori fillings. Restricting to the use of torus knots as the pattern, we obtain a prior result for cabling operations [9, 11].

I am currently working on a project, joint with Krcatovich, to prove that one can obtain more L-space satellite operations by choosing a pattern from the list of L-space knots in [33]. Our approach would require an explicit computation of the knot Floer complex associated to such satellite knots, using techniques from bordered Heegaard Floer homology [18].

2. Results

2.1 Heegaard Floer homology and L-space knots

Heegaard Floer theory consists of a set of invariants of three- and four-dimensional manifolds [25]. For Y a rational homology three-sphere, one example of such invariants is $\widehat{HF}(Y)$, which is an abelian group satisfying rk $\widehat{HF}(Y) \geqslant |H_1(Y;\mathbb{Z})|$ [26]. By definition, Y is called an *L-space* whenever equality is achieved. Lens spaces, $L(p,q)$, form a large family of L-spaces. This fact can be seen by examining the Heegaard Floer complex associated to a genus one Heegaard splitting of $L(p,q)$. A knot $K \subset S^3$ that admits an L-space Dehn surgery is called an *L-space knot*.

Torus knots are well-known to admit lens space surgeries. A generalization is the family of *twisted torus knots*, $K(p,q;s,r)$, which are defined to be (p,q) torus knots with r full twist(s) on s adjacent strands. In [33] I classify the L-space twisted $(p, kp\pm 1)$ torus knots. For $p,k,s,r > 0$:

Theorem 0.1. *The twisted torus knot, $K(p,kp\pm 1;s,r)$, is an L-space knot if and only if either $s = p-1$ or $s \in \{2, p-2\}$ and $r = 1$.*

A key ingredient of the proof is the observation that all of the twisted $(p, kp\pm 1)$ torus knots are $(1,1)$ *knots*, i.e. knots that can be placed in one-bridge position with respect to a genus one Heegaard splitting of S^3. From the perspective of knot Floer homology, $(1,1)$ knots are particularly appealing, since they can presented by a *doubly-pointed Heegaard diagram* of genus one [7]. The chain complex for knot Floer homology is defined in terms of a doubly-pointed Heegaard diagram. As shown by Ozsváth and Szabó [24], for knots admitting a genus one Heegaard diagram, knot Floer homology can be computed combinatorially and efficiently.

Another goal in the study of L-space knots is to classify the satellite operations on knots that produce L-space knots. Recall that, the construction of a satellite knot involves a knot K in S^3 and a knot P in $V = S^1 \times D^2$. The satellite knot, $P(K)$, with companion K and pattern P, is the image of P under an embedding $f: V \to S^3$ which maps V to a regular neighborhood of K. By combining work of Hedden [9] and Hom [11], the (m,n) cable of a knot $K \subset S^3$ is an L-space knot if and only if K is an L-space knot and $n/m \geqslant 2g(K)-1$. Hom, Lidman and I generalize this result in [12] by introducing new L-space satellite operations using Berge-Gabai knots [6] as the patterns. To see this as a generalization, it should be noted that any torus knot is a Berge-Gabai knot [20].

It is shown in [6] that any Berge-Gabai knot must be either a torus knot or a *1-bridge braid* in $S^1 \times D^2$. A Berge-Gabai knot P in $V = S^1 \times D^2$ with winding number w arises from the following construction. In the braid group B_w let σ_i denote the generator of B_w that performs a positive half twist on strands i, $i+1$. Let $\sigma = \sigma_b\sigma_{b-1}...\sigma_1$ be a braid in B_w with $0 \leqslant b \leqslant w-2$ and let t be an integer satisfying $1 \leqslant t \leqslant w-2$. Place σ in a solid cylinder and glue the ends so that the bottom of the strands are connected to the top after a $2\pi t/w$ positive twist. This construction forms a torus knot if $b=0$ and a 1-bridge representation of P in V if $1 \leqslant b \leqslant w-2$. We call b the *bridge width* and t the *twist number* of P. Our result states that:

Theorem 0.2. *Let P be a $q \geqslant 0$ times positively twisted Berge-Gabai knot with bridge width b, twist number t, and winding number w, and let K be an arbitrary knot in S^3. Then the satellite $P(K)$ is an L-space knot if and only if K is an L-space knot and $\dfrac{b+tw+qw^2}{w^2} \geqslant 2g(K)-1$.*

Note that when $b = 0$, because we can take $m = w$ and $n = t + qw$, Theorem 0.2 reduces to the cabling result of [11]. By applying techniques developed in [6, 8] to carefully explore the framing change of the solid torus surgered along P, we prove the "if" direction of the theorem. More precisely, surgery on $P(K)$ corresponds to first doing surgery on P, and attaching this to the exterior of K. Therefore, if one chooses the filling on P such that the result is a solid torus (using that P is a Berge-Gabai knot), then the induced surgery on $P(K)$ corresponds to attaching a solid torus to the exterior of K (performing surgery on K). Moreover, note that by adding q positive Dehn twists to a Berge-Gabai knot P, we can obtain an infinite family of Berge-Gabai knots. Fixing an L-space knot K, as q increases, the satellite switches from a knot with no L-space surgeries to an L-space knot. The latter fact holds since:

Remark 0.3. *For an L-space knot K, $S^3_r(K)$ is an L-space whenever $r \geqslant 2g(K) - 1$.*

Finally, the "only if" direction is proved by equating the ranks of $\widehat{HF}$ for the manifolds obtained from surgeries on K and $P(K)$.

Our result can be applied to produce an infinite family of manifolds with "nice" JSJ decompositions. Recall that irreducible, orientable, closed three-manifolds have a unique (up to isotopy) minimal collection of disjointly embedded incompressible tori such that each component of the three-manifold obtained by cutting along the tori is either atoroidal or Seifert-fibered. Moreover, closed atoroidal Haken manifolds are hyperbolic [30]. Theorem 0.2 can be used to obtain L-spaces with arbitrary hyperbolic and Seifert fibered pieces in their JSJ decompositions.

Corollary 0.4. *For non-negative integers m and n, there exist an infinite family of L-spaces such that their JSJ decompositions consist of m hyperbolic pieces and n Seifert fibered pieces.*

2.2 Sutured Floer homology and Seifert surfaces

András Juhász in 2006, introduced sutured Floer homology (denoted SFH) which is an invariant of three-manifolds with boundary together with a collection of curves in the boundary satisfying certain types of conditions. He posed the question, whether or not SFH can be used to distinguish two minimal genus Seifert surfaces of a given knot $K \subset S^3$. Technically, a knot K can have more than one minimal genus Seifert surface, and two Seifert surfaces for K are considered to be equivalent if there is an isotopy of S^3 taking one surface to the other. Fiberedness of a knot is known as a sufficient condition for which its minimal genus Seifert surface is unique (see [3]). However, there are many known examples of knots with non-isotopic minimal genus Seifert surfaces. See for instance [1, 10, 15, 16, 19, 31]. In [32] I find a family of knots with trivial Alexander polynomial with two minimal genus Seifert surfaces for each member in the family. This construction provides the first use of sutured Floer homology and not merely its Euler characteristic (a classical torsion) to distinguish the Seifert surfaces. The latter fact follows from the triviality of the Alexander polynomials.

Theorem 0.5. *Let $P(K_1, K_2)$ be the knot obtained by plumbing two knotted annuli with arbitrary knots K_1 and K_2 as in Figure 1, with framings l and 0, respectively, $l \neq 0$. Changing the plumbing results in the same knot, but two inequivalent Seifert surfaces, R and R'.*

The technique to prove this theorem, begins by noting that the surfaces' complements have a particular structure called a sutured manifold [5]. One cannot possibly use only the rank of SFH of minimal genus Seifert surfaces' complements to distinguish them, since the rank in this

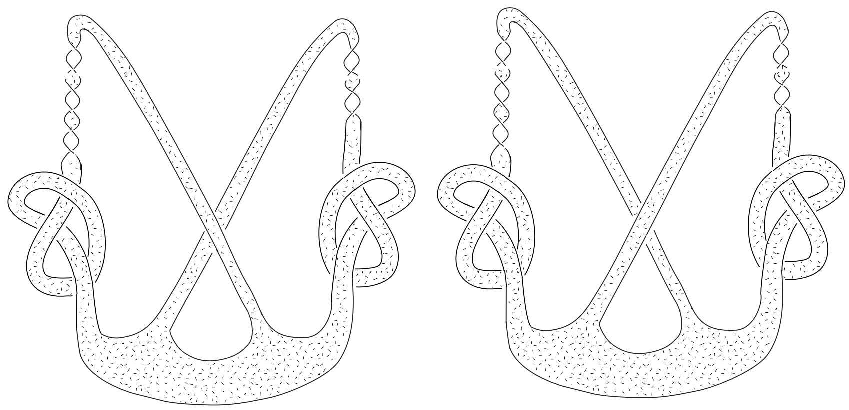

FIGURE 1. The above pictures are over/under plumbings of two twisted annuli, R and $R^{'}$ respectively, both are bounded by the same knot, $P(K_1, K_2)$, where K_1 is the right handed trefoil and K_2 is the left handed trefoil. These lead to two distinguished Seifert surfaces R and $R^{'}$, up to equivalence, for the knot $P(K_1, K_2)$.

case depends only on the knot [14, Theorem 1.5]. Therefore, we need to know the structure of SFH as a Spin^c-graded group if we are able to use it to show that the two surfaces are not equivalent. Combining the sutured Floer homology of surfaces' complements with the Seifert form, turns out to be a useful tool in distinguishing different Seifert surfaces [10].

The reason I was interested in the particular knots is twofold. First, classical methods (considering the invariants of the homeomorphism type of the surfaces' complements, e.g the fundamental group) fail in distinguishing the two Seifert surfaces. Second, they have isomorphic SFH groups.

3. Current Research

3.1 L-spaces and left-orderability

The overarching goal of my recent results [12, 33] is to look for possible characterizations of L-spaces that do not reference Heegaard Floer homology. Examples of L-spaces include lens spaces and all connected sums of manifolds with elliptic geometry. These examples also have the property that their fundamental groups do not admit any *left ordering*.

Definition 0.6. *A non-trivial group G is called* left-orderable *if there exists a strict total ordering $<$ on its elements such that $g < h$ implies $fg < fh$ for all elements $f, g, h \in G$.*

The following conjecture for all L-spaces was made in [2]:

Conjecture 0.7. *A rational homology three-sphere, Y, is an L-space if and only if $\pi_1(Y)$ is not left-orderable.*

In the light of this conjecture and Theorem 0.2, Hom, Lidman and I pose another conjecture:

Conjecture 0.8. *Let $P(K) \subset S^3$ denote the satellite knot with companion K and pattern P. If $P(K)$ has a non-left-orderable surgery, then both P and K admit non-left-orderable surgeries.*

At this point, we can prove that P admits a non-left-orderable surgery when P is a Berge-Gabai knot [4]. Recall that L-spaces do not admit taut foliations [23]. It is natural to wonder if L-spaces can be characterized as those closed, connected three-manifolds that admit no taut foliations:

Conjecture 0.9. *A rational homology three-sphere, Y, is an L-space if and only if Y admits no taut foliations.*

4. Future Research

In ongoing work with David Krcatovich, I hope to produce more L-space satellite operations, choosing the patterns from the list of L-space twisted torus knots in [33]. Our strategy is to explicitly compute the knot Floer complex associated to such satellite knots, using the genus one Heegaard diagrams constructed for the L-space twisted $(p, kp \pm 1)$ torus knots together with techniques from bordered Heegaard Floer homology.

Another aspect of the Ozsváth-Szabó invariants that intrigues me is their relationship to foliations. Some of the most powerful results in Heegaard Floer theory are related to the existence of certain foliations on three-manifolds. I am particularly interested in classifying taut foliations on the complements of L-space knots. More precisely, it is known that if K is an L-space knot, then $S^3_r(K)$ possesses no taut foliation whenever $r \geqslant 2g(K) - 1$. Therefore, $S^3 - n\overset{\circ}{b}(K)$ admits no taut foliation with lines of slope $r \geqslant 2g(K) - 1$ with $r \in \mathbb{Q}$, else a Dehn filling with coefficient r extends the taut foliation of $S^3 - n\overset{\circ}{b}(K)$ to $S^3_r(K)$. In the light of Conjecture 0.9 and Remark 0.3, for K an L-space knot, it seems reasonable to expect taut foliations on $S^3_r(K)$ whenever $r < 2g(K) - 1$. The case for the right handed trefoil is adduced in [29] where the foliations are constructed on the complement, with slope of the lines of the foliation on the boundary varying in the interval $(-\infty, 1)$. Similar techniques have been used for the case of torus knots in [21]. Building on these works, it would be useful to apply these constructive techniques to the L-space twisted torus knots of [33].

Another area of low dimensional topology where the Ozsváth-Szabó invariants have had a significant impact is in the study of Dehn surgery. One example is the cosmetic surgery problem:

Conjecture 0.10. *Two surgeries on a nontrivial knot with non-equivalent slopes are never homeomorphic as oriented manifolds.*

For knots in S^3, Ni and Wu reduce the conjecture to case where the slopes are negatives of each other [22]. Their result draws on the close relationship between the knot Floer homology invariants of a knot K and the Ozsváth-Szabó invariants of $S^3_r(K)$. Is it possible to extend this result and prove that these slopes must be trivial? Is it possible to obtain a similar obstruction when K is a knot in a rational homology three-sphere and not necessarily in S^3? Also note that two homeomorphic three-manifolds must have isomorphic *linking forms* where the linking form of a three manifold Y is the non-degenerate form

$$\Phi_M : Tor_Y \otimes Tor_Y \to \mathbb{Q}/\mathbb{Z}$$

on the torsion subgroup Tor_Y of $H_1(Y;\mathbb{Z})$ defined by $\Phi_Y(a \otimes b) = \alpha \cdot \tau/n$. Note that α is any 1-cycle representing a and τ is any 2-chain bounded by a positive integral multiple $n\beta$ of a 1-cycle β representing b. When do two different surgeries on a knot result in equivalent linking forms? These are all questions I would like to explore.

References

[1] William Robert Alford. Complements of minimal spanning surfaces of knots are not unique. *Ann. of Math.*, 91(2):419–424, 1970.

[2] Steven Boyer, Cameron McA. Gordon, and Liam Watson. On L-spaces and left-orderable fundamental groups. *Mathematische Annalen*, 356(4):1213–1245, December 2012.

[3] Gerhard Burde and Heiner Zieschang. Neuwirthsche Knoten und Flachenabbildungen. *Abh. Math. Sem. Univ. Hamburg*, 31:236–249, 1967.

[4] Adam Clay, Tye Lidman, and Liam Watson. Graph manifolds, left-orderability and amalgamation. *Algeb. Geom. Topol.*, 13(4):2347–2368, July 2013.

[5] David Gabai. Foliations and the topology of 3-manifolds. *J. Differential Geom.*, 18:445–503, 1983.

[6] David Gabai. 1-bridge braids in solid tori. *Topology and its Applications*, 37:221–235, 1990.

[7] Hiroshi Goda, Hiroshi Matsuda, and Takayuki Morifuji. Knot Floer homology of $(1,1)$-knots. *Geometriae Dedicata*, 112(1):197–214, April 2005.

[8] Cameron McA. Gordon. Dehn Surgery and satellite knots. *Transactions of the American Mathematical Society*, 275(2):687, February 1983.

[9] Matthew Hedden. On knot Floer homology and cabling II. *Int. Math. Res. Not. IMRN*, 12:2248–2274, 2009.

[10] Matthew Hedden, András Juhász, and Sucharit Sarkar. On sutured Floer homology and the equivalence of Seifert surfaces. *arXiv:0811.0178*, pages 1–32, 2008.

[11] Jennifer Hom. A note on cabling and L-space surgeries. *Algeb. Geom. Topol.*, 11(1):219–223, January 2011.

[12] Jennifer Hom, Tye Lidman, and Faramarz Vafaee. Berge-Gabai knots and L-space satellite operations. *In prepration.*

[13] András Juhász. Holomorphic discs and sutured manifolds. *Algeb. Geom. Topol.*, 6:1429–1457, October 2006.

[14] András Juhász. Floer homology and surface decompositions. *Geom. Topol.*, 12(1):299–350, March 2008.

[15] Osamu Kakimizu. Classification of the incompressible spanning surfaces for prime knots of 10 or less crossings. *Hiroshima Math. J.*, 35:47–92, 2005.

[16] Tsuyosht Kobayashi. Uniqueness of minimal genus Seifert surfaces for links. *Topology Appl.*, 33:265–279, 1989.

[17] Tye Lidman and Allison Moore. Pretzel knots with l-space surgeries. *arXiv:1306.6707*, pages 1–24, 2013.

[18] Robert Lipshitz, Peter Ozsváth, and Dylan Thurston. Bordered Heegaard Floer homology. *arXiv:0810.0687v4*, pages 1–271, 2011.

[19] Herbert C Lyon. Simple knots without unique minimal surfaces. *A.M.S.*, 43(2):449–454, 1974.

[20] Louise Moser. Elementary surgery along a torus knot. *Pacific Journal of Mathematics*, 38(3):737–745, September 1971.

[21] Yasuharu Nakae. Taut Foliations of Torus Knot Complements. *J. Math. Sci. Univ. Tokyo*, 14:31–67, 2007.

[22] Yi Ni and Zhongtao Wu. Cosmetic surgeries on knots in S^3. *arXiv: 1009.4720*, pages 1–16, 2010.

[23] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and genus bounds. *Geom. Topol.*, 8:311–334, 2004.

[24] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and knot invariants. *Adv. Math.*, 186(1):58–116, August 2004.

[25] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and three-manifold invariants: properties and applications. *Ann. of Math.*, 159(3):1159–1245, 2004.

[26] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math.*, 159(3):1027–1158, 2004.

[27] Peter Ozsváth and Zoltán Szabó. On knot Floer homology and lens space surgeries. *Topology*, 6(44):1281–1300, 2005.

[28] Jacob Rasmussen. Floer homology and knot complements. *PhD Thesis, Harvard University*, 2003.

[29] Rachel Roberts. Taut foliations in punctured surface bundles I. *Proceedings of London Mathematical Society*, 82(3):747–768, 2001.

[30] William P. Thurston. Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bulletin of the A.M.S.*, 6(3):357–382, May 1982.

[31] H. F. Trotter. Some knots spanned by more than one knotted surface of mininal genus. *Princeton Univ. Press*, page 51.62, 1975.

[32] Faramarz Vafaee. Seifert surfaces distinguished by sutured Floer homology but not its Euler characteristic. *arXiv:1204.2452v1*, pages 1–19, 2012.

[33] Faramarz Vafaee. On the knot Floer homology of twisted torus knots. *arXiv:1311.3711v1*, pages 1–17, 2013.

[34] Liam Watson. A surgical perspective on quasi-alternating links. *arXiv:0910.0449v1*, pages 1–16, 2009.