Models of Dependencies for Assessing Solvency

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Solvency defined

Solvency status of a company is assessed at a particular period requiring sufficient capital is held to cover expected liabilities over a fixed time horizon, with a high degree of probability confidence.

Technically, if S is the aggregated random loss over the time horizon, the solvency capital requirement (SCR), term used in Sandström (2006), is

$$\text{SCR}_S = \rho(S) - \mathbb{E}(S),$$

where ρ is a risk measure defined to be a mapping from set Γ of real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the real line $\mathbb{R}$:

$$\rho : \Gamma \rightarrow \mathbb{R} : S \in \Gamma \rightarrow \rho(S).$$

Risk measures - Artzner (1999).

The aggregation of risks

The company's random loss S is usually the sum of several components

$$S = X_1 + X_2 + \cdots + X_n,$$

where the components $X_1, X_2, \ldots, X_n$ can be interpreted as:

- the individual losses corresponding to the losses of the several business units within the company;
- the individual losses arising from the different policies within the company's portfolio of policies; or
- the individual losses arising from various categories of risks such as the underwriting, credit, market and operational risks.

Popular risk measures

Premium principles are clear examples of risk measures. Goovaerts (1984).

Risk measures must be practically simple to calculate and easily understood.

Two widely known and used risk measures are:

- Value-at-Risk (VaR): For $0 < q < 1$, the q-th quantile risk measure is defined to be

$$\text{VaR}_q(S) = \inf(s | F_S(s) \geq q).$$

- Tail Value-at-risk: The Tail VaR is defined to be

$$\text{TVaR}_q(S) = \mathbb{E}(S | S > \text{VaR}_q(S)).$$

Both risk measures are used in several regulatory regimes as well as by rating agencies such as Standard & Poor's.

Possible effect of risk interactions

To determine solvency capital, convention is:

- first identify various sources of risks;
- quantify these risks (with probabilistic models);
- determine separate amount of capital needed for each risk; and
- account for possible interaction of risks which may lead to possible diversification effect.

Typically, diversification is interpreted so that this leads to some form of a benefit:

$$\mathsf{SCR}_S \leq \mathsf{SCR}_{X_1} + \cdots + \mathsf{SCR}_{X_n}.$$

Because expectation is a linear operator, this leads us to a choice of a subadditive risk measure:

$$\rho(S) \leq \rho(X_1) + \cdots + \rho(X_n).$$

The classification of risks

A typical insurer would classify risks according to:

- Asset default risk - potential losses arising from investment default.
- Interest rate risk - risk of losses because of changes in the level of interest rates causing a mismatch in asset and liability cash flows.
- Credit risk - risk arising from inability to recover from reinsurers or other sources of risk transfer arrangements.
- Underwriting risk - risk of losses arising from excess claims (pure random fluctuations or prediction inaccuracies).
- Other business risk - the “catch-all-else” category including e.g. operational losses.

Risk-based capital models

Most risk-based capital (RBC) models attempt to quantify capital requirements according to the company's exposure to risks.

- These are formula-based in the sense that for each sources of "quantifiable" risk, a set of factors (or percentages) are recommended to establish a set of *Minimum Capital Requirements*.
- This approach has been recommended by the National Association of Insurance Commissioners (NAIC) in the United States since the 1990's, and has been the model followed even till today.
- The NAIC formula-based capital requirement has been similarly adopted by rating agencies such as:
  - Standard & Poor's; and
  - A.M. Best.

Comparing risk-based capital charges

The case of general insurers

Risk categories	NAIC	S & P	A.M. Best
Asset risk charges:			
Bonds	0 - 30%	0 - 30%	0 - 30%
Common Stock	20 - 43%	15%	15%
Real Estate	18 - 29%	10%	20%
Credit risk charges:		vary by	vary by
Reinsurance recoverables	10%	reinsurer's rating	reinsurer's rating
Written premium risk charges:			
Homeowners	vary by line of	21 - 35%	37 - 54%
Other liability occurrence	business with initial	30 - 49%	32 - 40%
CMP	industry factor	13 - 21%	29 - 37%
Personal auto	adjusted for company	9 - 14%	25 - 40%
Property	experience	9 - 14%	33 - 51%
Reserve risk charges:			
Homeowners	vary by line of	11 - 19%	19 - 39%
Other liability occurrence	business with initial	14 - 23%	26 - 48%
CMP	industry factor	5 - 9%	25 - 45%
Personal auto	adjusted for company	10 - 16%	20 - 48%
Property	experience	28 - 46%	26 - 47%

Source: M. Carrier, Deloitte Consulting LLP, Risk-Based Capital: So Many Models, presentation slides at the CAS Annual Meeting 2007.

Solvency II

Solvency II is a by-product of the European Commission to develop new solvency system of regulatory requirements for insurers to operate in the European Union.

- Framework somewhat patterned after the New Basel Capital Accord (Basel II) on banking supervision.
- To achieve some sort of uniformity in regulations for establishing capital.
- Based on broad "risk-based" principles in the measurement of assets and liabilities.
- The primary aims are:
  - to reduce the probability of insolvency; and
  - if it does occur, to reduce the financial and economic impact to affected policyholders.

Solvency II framework

Solvency II framework consists of 3 pillars.

- Pillar 1 - consists of identifying the risks and quantifying the amount of capital required.
  - fair valuation of assets/liabilities;
  - some prescription of factor-based methods to calculate minimum capital; but
  - use of internal models allowed, provided justified.
- Pillar 2 - prescribes requirement for effective risk management systems and processes with steps towards effective supervisory review and examination.
- Pillar 3 - focuses on a more discipline in the market including fair disclosure and more transparency.

Additional details can be found in: www.fsa.gov.uk

The case of Brazil

We are seeing some significant changes in the insurance market e.g. reinsurance introduced, concurrent development in insurance supervision and regulation.

- Inspired by the Solvency II framework, new capital requirements, effective early 2008, for the insurance industry were being established and implemented.
- New requirements were released as two guidelines (Resolutions No. 155 and 158), developed by the National Private Insurance Council (CNSP - *Conselho Nacional de Seguros Privados*), a governing body responsible for insurance policies in Brazil, and its executive body, the Superintendent of Private Insurance (SUSEP).

Old vs new requirements

Old and new requirements:

- Old requirement: determined according to the company's mix of product lines and according to the geographical regions in which the company is authorized to conduct business.

- New requirement: the minimum capital to be based on the sum of a "base capital" and an "additional capital", which have to be continually maintained to be allowed to continue operating as an insurer.

Resolutions 155 and 158

Resolution 155:

- Establishes definition of the "base capital" (a fixed amount according to region), and the "additional capital" (a variable component reflecting the risks categorized according to credit, market, underwriting, legal and operational risks).
- Gives further details of the required action should there be capital inadequacy.

Resolution 158:

- Inspired by Solvency II, permits insurers to develop own internal capital models which could allow holding lower capital.
- Must demonstrate compliance; required to submit balance sheet statements to SUSEP every 6 months and depending on degree of non-compliance, penalties imposed.

Additional details can be found in Sommer (May 2007) and Sommer (March 2008).

Notions of multivariate modeling

Let $\mathbf{X} = (X_1, \ldots, X_n)'$ be an n-dimensional random vector.

Joint distribution. Its joint distribution function is

$$F(\mathbf{x}) = F(x_1, \ldots, x_n) = \mathrm{P}(X_1 \leq x_1, \ldots, X_n \leq x_n).$$

Marginals. The marginals of the individual components are

$$F_i(x_i) = \mathrm{P}(X_i \leq x_i) = F(\infty, \ldots, \infty, x_i, \infty, \ldots, \infty).$$

Densities. If the density exist, it can be derived from

$$f(x_1, \ldots, x_n) = \frac{\partial^n F(x_1, \ldots, x_n)}{\partial x_1 \cdots \partial x_n}$$

and is related to the joint distribution by

$$F(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(u_1, \ldots, u_n) du_1 \cdots du_n.$$

Covariance and correlation

For a random vector $\mathbf{X}$, the covariance matrix is defined by

$$\mathsf{Cov}(\mathbf{X}) = \mathsf{E}((\mathbf{X} - \mathsf{E}(\mathbf{X}))(\mathbf{X} - \mathsf{E}(\mathbf{X}))'),$$

assuming they exist.

Often, we write $\boldsymbol{\Sigma}$ to denote this covariance matrix with the (i, j)th element expressed as

$$\sigma_{ij} = \mathsf{Cov}(X_i, X_j) = \mathsf{E}(X_i X_j) - \mathsf{E}(X_i)\mathsf{E}(X_j).$$

The correlation matrix, denote it by $\mathbf{R}$, has (i, j)th element equal to

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}},$$

the ordinary pairwise linear correlation of X_i and X_j. If we write $\Delta = \mathsf{diag}(\sqrt{\sigma_{11}}, \ldots, \sqrt{\sigma_{nn}})$, then we have $\mathbf{R} = \Delta^{-1}\Sigma\Delta^{-1}$.

Some important properties

Given a matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{a} \in \mathbb{R}^m$, covariance is

$$\text{Cov}(A\mathbf{X} + \mathbf{a}) = A\text{Cov}(\mathbf{X})A'.$$

For linear combinations of the components of $\mathbf{X}$, we therefore find that

$$\text{Var}(\mathbf{a}'\mathbf{X}) = \mathbf{a}'\Sigma\mathbf{a},$$

for any vector $\mathbf{a} \in \mathbb{R}^n$. It follows that this variance is usually non-negative because covariance matrices must be positive semi-definite.

Independence

The components of $\mathbf{X}$ are mutually independent if and only if

$$F(\mathbf{x}) = \prod_{k=1}^{n} F_k(x_k), \quad \text{for all } \mathbf{x} \in \mathbb{R}^n,$$

or, if the densities exist, if and only if

$$f(\mathbf{x}) = \prod_{k=1}^{n} f_k(x_k), \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Correlation and independence. Independence implies zero covariance and hence zero correlation, but the converse is not necessarily true.

However, the converse is true for the case of multivariate normal distributions.

The multivariate Normal distribution

$\mathbf{X} \sim \mathrm{N}_n(\mu, \mathbf{\Sigma})$ if its joint density is

$$f(\mathbf{x}) = (2\pi)^{-n/2} |\mathbf{\Sigma}|^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{x}-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right].$$

Mean vector is μ and covariance matrix is $\mathbf{\Sigma}$.

The components of $\mathbf{X}$ are mutually independent if and only if the covariance is a diagonal matrix.

The standard multivariate normal is the case where $\mathbf{X} \sim \mathrm{N}_n(\mathbf{0}, \mathbf{I})$ where $\mathbf{I}$ is an $n \times n$ identity matrix.

We often write the vector $\mathbf{Z}$ to denote standard multivariate normal.

Some problems with multivariate normal

Some believe that there are deficiencies of the normal for multivariate modeling in finance/insurance:

- The tails of the margins may be too thin, and hence fail to generate some extreme values.
- As a consequence, in the multivariate sense, it fails to capture phenomenon of joint extreme movements. Simultaneous large values may be relatively infrequent - generally believed to lack tail dependence.
- Too much symmetry - lack of presence of skewness. Some financial/insurance data exhibits long tails.

The multivariate t distribution

$\mathbf{X} \sim \mathrm{t}_n(\nu, \mu, \mathbf{\Sigma})$ if its joint density is

$$f(\mathbf{x}) = \frac{\Gamma((\nu+n)/2)}{(\pi\nu)^{n/2}\Gamma(\nu/2)|\mathbf{\Sigma}|^{1/2}} \left[1 + \frac{1}{\nu}(\mathbf{x}-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right]^{-(\nu+n)/2}.$$

Mean vector is μ and covariance is $\mathrm{Cov}(\mathbf{X}) = \frac{\nu}{\nu-2}\mathbf{\Sigma}$.

In the case where $\mathbf{\Sigma}$ is diagonal, then the components of $\mathbf{X}$ are uncorrelated; however, they are not independent.

The multivariate t has heavier tails than the multivariate normal.

If $\mathbf{Y} \sim \mathrm{N}_n(\mathbf{0}, \mathbf{\Sigma})$ and if $\nu S^2/\sigma^2 \sim \chi^2_\nu$ distribution, then the multivariate t random variable has the representation

$$\mathbf{X} = S^{-1}\mathbf{Y} + \mu.$$

As $\nu \to \infty$, the multivariate t approaches the multivariate normal.

Multivariate distribution function

A function $F : \mathbb{R}^n \to [0,1]$ is a multivariate distribution function if it satisfies:

- right-continuity;
- $\lim_{x_i \to -\infty} F(x_1, \ldots, x_n) = 0$ for $i = 1, \ldots, n$;
- $\lim_{x_i \to \infty, \forall i} F(x_1, \ldots, x_n) = 1$; and
- rectangle inequality holds: for all $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ with $a_i \leq b_i$ for $i = 1, \ldots, n$, we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} F(x_{1i_1}, \ldots, x_{ni_n}) \geq 0,$$

  where $x_{i1} = a_i$ and $x_{i2} = b_i$.

Copula defined

A copula $C : [0,1]^n \rightarrow [0,1]$ is a multivariate distribution function whose univariate marginals are Uniform$(0,1)$.

Properties of a copula:

- $C(u_1, \ldots, u_n)$ must be increasing in each component u_i.
- $C(u_1, \ldots, u_{k-1}, 0, u_{k+1}, \ldots, u_n) = 0$.
- $C(1, \ldots, 1, u_k, 1, \ldots, 1) = u_k$.
- the rectangle inequality leads us to

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{1i_1}, \ldots, u_{ni_n}) \geq 0$$

for all $u_i \in [0,1]$, $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ with $a_i \leq b_i$, and $u_{i1} = a_i$ and $u_{i2} = b_i$.

Sklar's representation theorem

Sklar (1959): There exists a copula function C such that

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

where F_i is the marginal for X_i, $i = 1, \ldots, n$.

Equivalently, we write

$$\mathsf{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) = C(\mathsf{P}(X_1 \leq x_1), \ldots, \mathsf{P}(X_n \leq x_n)).$$

C need not be unique, but it is unique for continuous marginals. Else, C is uniquely determined on $\text{Ran}F_1 \times \ldots \times \text{Ran}F_n$.

In the continuous case, this unique copula can be expressed as

$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)),$$

where F_i^{-1} are the respective quantile functions.

Some examples

Independence copula: $C(u_1, \ldots, u_n) = u_1 \cdots u_n$.

The Fréchet bounds: Any copula function satisfies the following bounds:

$$L_F(u_1, \ldots, u_n) \leq C(u_1, \ldots, u_n) \leq U_F(u_1, \ldots, u_n),$$

where

Fréchet lower bound: $L_F = \max(\sum_{i=1}^n u_i - (n-1), 0)$, and

Fréchet upper bound: $U_F = \min(u_1, \ldots, u_n)$.

The Fréchet upper bound (comonotonic copula) satisfies definition of a copula, but the Fréchet lower bound is a copula only in the case of $n = 2$ dimension (countermonotonic copula).

The comonotonic copula

Define the comonotonic copula $C_U = \min(u_1, \ldots, u_n)$.

It can be shown that if $F_1, \ldots, F_n$ are univariate marginal distribution functions, then C_U is the distribution function of the random vector

$$(F_1^{-1}(U), \ldots, F_n^{-1}(U)),$$

where F_i^{-1} are the usual quantile functions.

Comonotonicity is indeed a very strong *positive* dependency structure - results in very strong positive comovements. The higher the value of one component X_i, the higher the value of any other component X_j.

Studied by: Dhaene, et al. (2002a, 2002b). Very useful for finding bounds of functions of components of a random vector.

Invariance property

Suppose random vector $\mathbf{X}$ has copula C and suppose $T_1, \ldots, T_n$ are non-decreasing continuous functions of $X_1, \ldots, X_n$, respectively.

The random vector defined by $(T_1(X_1), \ldots, T_n(X_n))$ has the same copula C.

The usefulness of this property can be illustrated in many ways. If you have a copula describing joint distribution of insurance losses of various types, and you decide the quantity of interest is a transformation (e.g. logarithm) of these losses, then the multivariate distribution structure does not change.

Hence, the dependency structure is preserved. However, the marginals do change.

Examples of (implicit) copulas

Normal copula:

$$C^n_{\mathbf{R}}(\mathbf{u}) = \mathbf{\Phi}_{\mathbf{R}}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)),$$

where Φ is the cdf of standard univariate normal, $\mathbf{\Phi}_{\mathbf{R}}$ is the joint cdf of $\mathbf{X} \sim \mathsf{N}_n(\mathbf{0}, \mathbf{R})$ with $\mathbf{R}$, the correlation matrix.

The case where $\mathbf{R} = \mathbf{I}_n$ results in independence, and $\mathbf{R} = \mathbf{J}_n$ gives comonotonicity.

t copula:

$$C^n_{\nu,\mathbf{R}}(\mathbf{u}) = \mathbf{t}_{\nu,\mathbf{R}}(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_n)),$$

where t_ν is the cdf of standard univariate t, $\mathbf{t}_{\nu,\mathbf{R}}$ is the joint cdf of $\mathbf{X} \sim \mathsf{t}_n(\nu, \mathbf{0}, \mathbf{R})$ with $\mathbf{R}$, the correlation matrix.

The case where $\mathbf{R} = \mathbf{J}_n$ gives comonotonicity, but $\mathbf{R} = \mathbf{I}_n$ does not result in independence.

Simulating normal and t copulas

Although implicit in forms, these copulas are easy to simulate from.

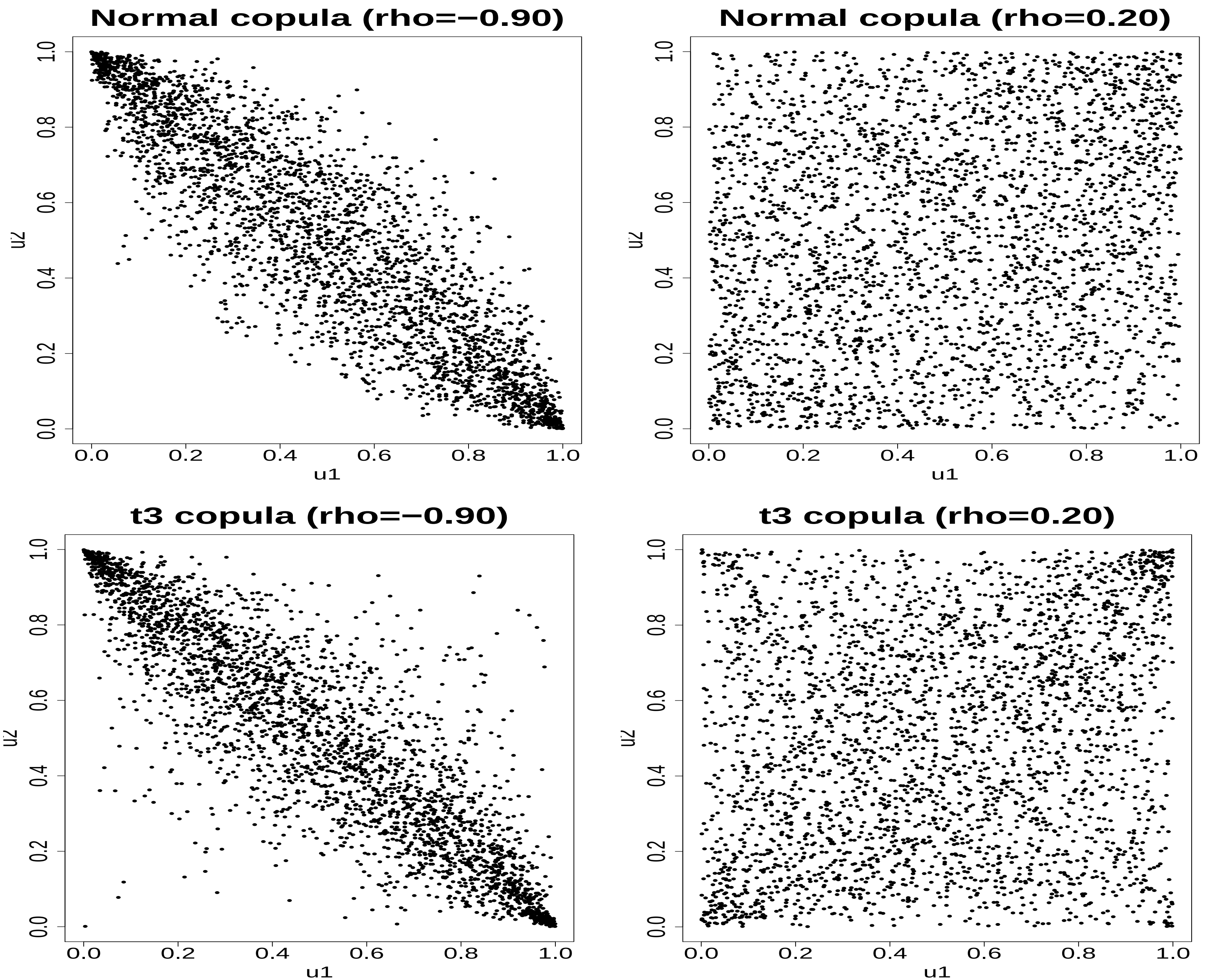
Simulating from normal copula:

1. simulate $\mathbf{X} \sim \mathrm{N}_n(\mathbf{0}, \mathbf{R})$;
2. set $\mathbf{U} = (\Phi(X_1), \ldots, \Phi(X_n))'$.

Simulating from t copula:

1. simulate $\mathbf{X} \sim \mathrm{t}_n(\nu, \mathbf{0}, \mathbf{R})$;
2. set $\mathbf{U} = (t_\nu(X_1), \ldots, t_\nu(X_n))'$.

Simulation - normal vs t copula



Special class: Archimedean copulas

C is an *Archimedean* if it has the form

$$C(u_1, \ldots, u_n) = \psi^{-1}(\psi(u_1) + \cdots + \psi(u_n)),$$

for some function ψ (called the generator) satisfying:

- $\psi(1) = 0$;
- ψ is decreasing; and
- ψ is convex.

To ensure you get a legitimate copula for higher dimensions, ψ^{-1} must be completely *monotonic*, i.e. its derivatives alternate in signs.

An important source of Archimedean generators is the inverses of the Laplace transforms of distribution functions.

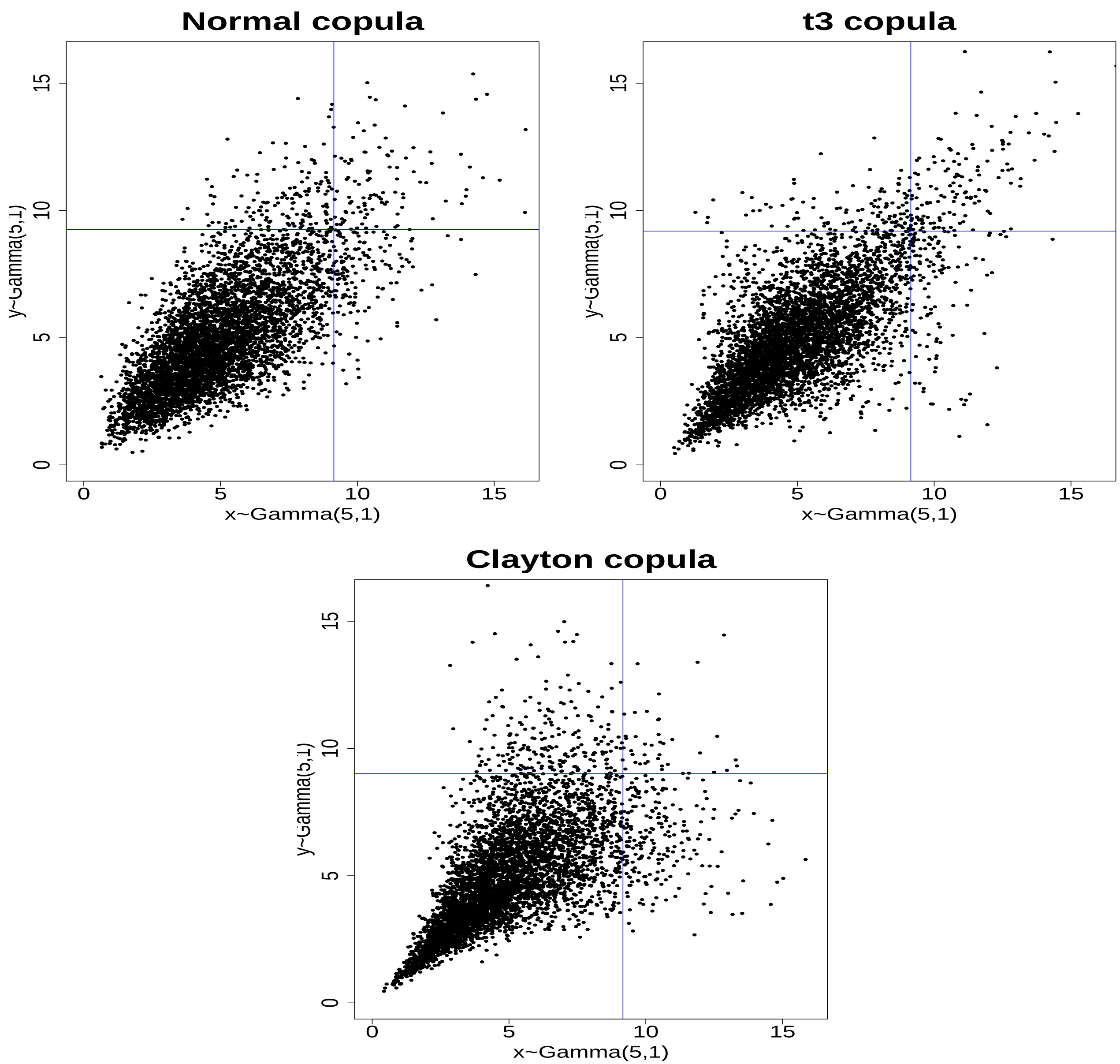
Feller (1971): A function φ on $[0, \infty]$ is the Laplace transform of a cdf F if and only if φ is completely monotonic with $\varphi(0) = 1$.

Archimedean copulas and their generators

Family	Generator $\psi(t)$	Range of α	$C(\mathbf{u})$
Independence	$-\log(t)$	na	$\prod_{i=1}^{n} u_i$
Clayton	$t^{-\alpha}-1$	$\alpha>0$	$\left[\sum_{i=1}^{n} u_i^{-\alpha}-n+1\right]^{-1/\alpha}$
Gumbel-Hougaard	$(-\log t)^{\alpha}$	$\alpha\geq 1$	$\exp\left\{-\left[\sum_{i=1}^{n}(-\log u_i)^{\alpha}\right]^{1/\alpha}\right\}$
Frank	$-\log\left(\frac{e^{-\alpha t}-1}{e^{-\alpha}-1}\right)$	$\alpha>0$	$-\frac{1}{\alpha}\log\left[1+\frac{\prod_{i=1}^{n}(e^{-\alpha u_i}-1)}{(e^{-\alpha}-1)^{n-1}}\right]$

Normal, t, and Clayton copulas

Marginals: Gamma(5,1), $\rho = 0.75$, and $\nu = 3$



Calibrating copula models

In demonstrating how to calibrate copula models, we consider empirical data with:

- Danish fire data provided by Mette Rytgaard.
- The data consists of 2,167 fire losses in Denmark for the period 1980-1990.
- The loss amounts vary according to:
  - buildings X_1
  - contents X_2
  - loss of profits X_3
- This same dataset has been used by Blum, Dias and Embrechts (2002), "The ART of Dependence Modelling", appearing in Alternative Risk Strategies, ed. M. Lane.

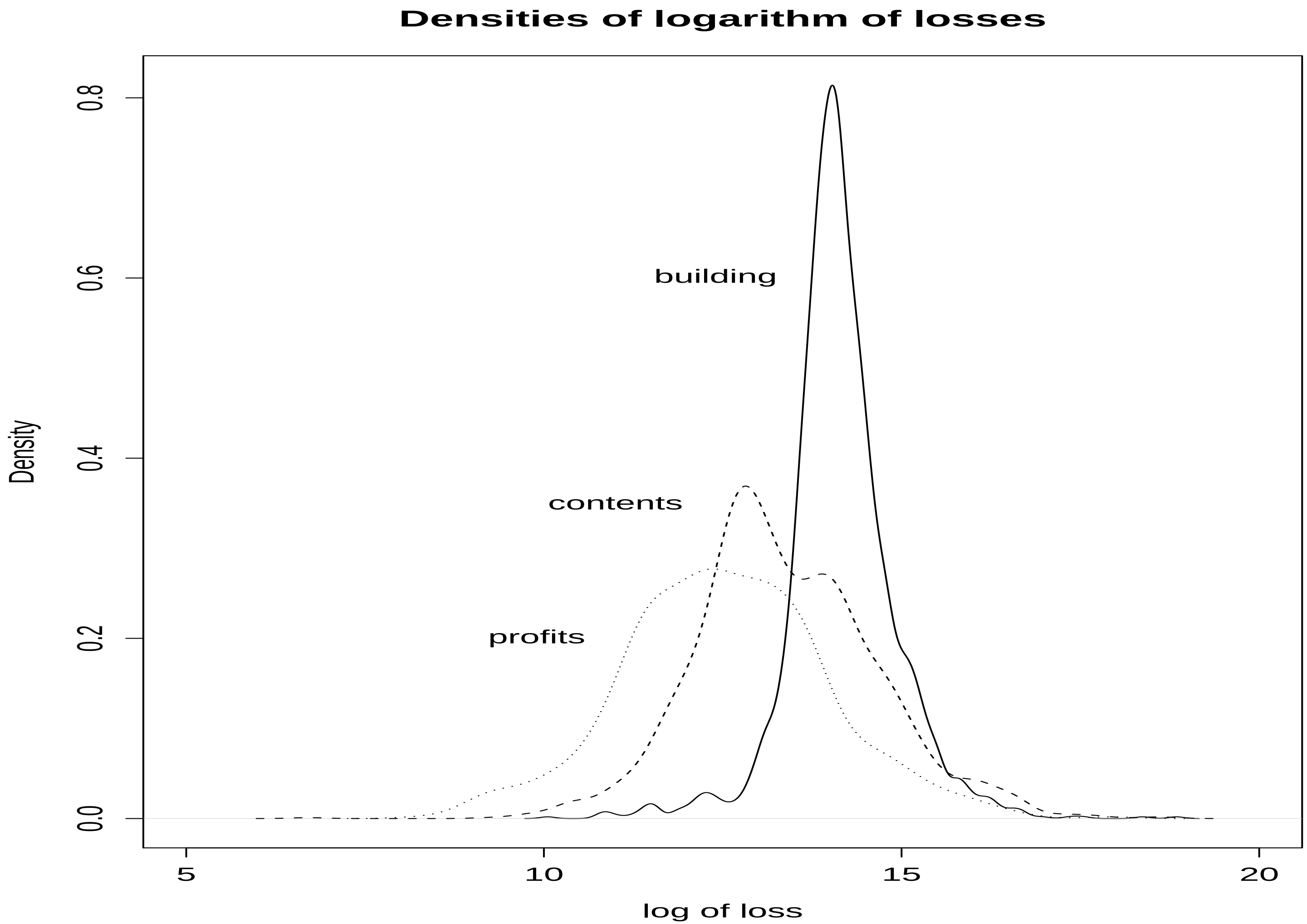
The first 10 observed values

building	contents	loss of profit	total
1.09809663	0.58565150	0.00000000	1.683748
1.75695461	0.33674960	0.00000000	2.093704
1.73258126	0.00000000	0.00000000	1.732581
0.00000000	1.30537600	0.47437775	1.779754
1.24450952	3.36749600	0.00000000	4.612006
4.45203953	4.27323400	0.00000000	8.725274
2.49487555	3.54319200	1.86090776	7.898975
0.77568960	0.99311710	0.43923865	2.208045
0.81259151	0.67349930	0.00000000	1.486091
2.37157394	0.16837480	0.25622255	2.796171

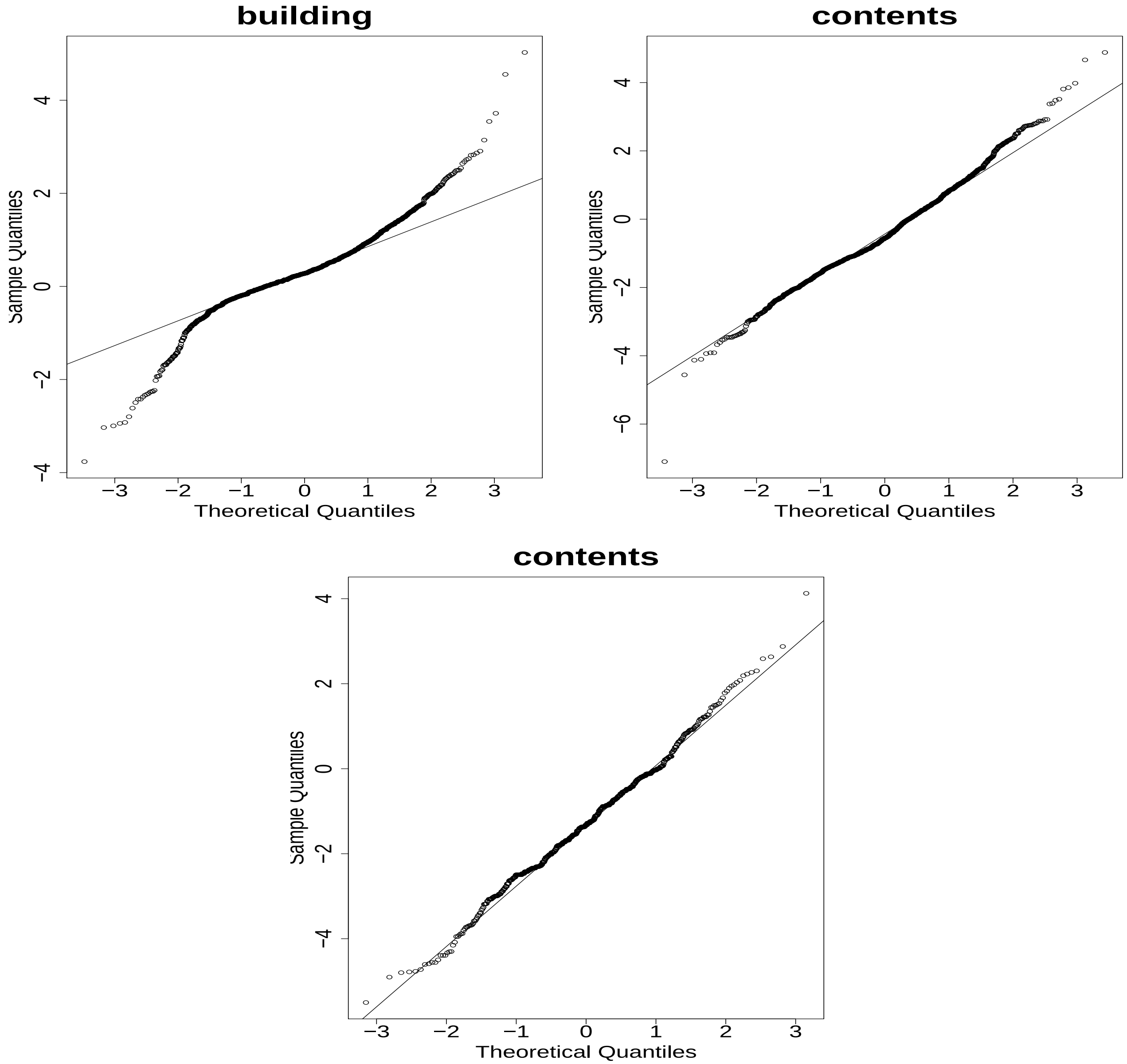
Some summary statistics

	type of loss			
	building	contents	loss of profit	total
zero counts	177	488	1,551	
(non-zero) count	1,990	1,679	616	2,167
mean	1,986,679	1,701,778	851,799	3,385,088
median	1,320,132	575,699	266,193	1,778,154
std dev	4,514,998	5,347,536	2,947,029	8,507,451
minimum	23,191	825	4,084	1,000,000
maximum	152,413,209	132,013,200	61,932,650	263,250,324
25th percentile	966,175	290,004	100,111	1,321,118
75th percentile	1,978,604	1,446,480	679,293	2,967,023

Marginal density plots



Q-Q plots of the logarithms



Fitting the marginals

To accomodate the large number of zeroes in each type of loss, we use a mixture model of the form:

$$f_k(x) = \begin{cases} p_k, & \text{for } x = 0 \\ (1 - p_k) f_{\mathsf{LN},k}(x), & \text{for } x > 0 \end{cases},$$

where $k = 1, 2, 3$ refers to the building, contents, and profits, respectively.

LN refers to the log-normal distribution with parameters μ and σ.

It is also easy to prove that the marginal CDF for the mixture is:

$$F_k(x) = p_k + (1 - p_k) F_{\mathsf{LN},k}(x), \text{ for } k = 1, 2, 3.$$

Marginal parameter estimates

Estimation used: Inference for Margins (IFM) method

Parameter	Building (X_1)	Contents (X_2)	Profits (X_3)
p	0.0817	0.2253	0.7156
(s.e.)	(0.0059)	(0.0090)	(0.0097)
μ	0.3384	-0.4257	-1.2802
(s.e.)	(0.0167)	(0.0310)	(0.0570)
σ	0.7438	1.2705	1.4153
(s.e.)	(0.0118)	(0.0219)	(0.0403)

Copula dependence parameter estimates

Parameter	Clayton copula	Normal copula	t-copula
α	0.0162		
(s.e.)	(0.0128)		
ρ_{BC}		0.3218	0.3194
(s.e.)		(0.0056)	(0.0017)
ρ_{BP}		0.2862	0.3005
(s.e.)		(0.0022)	(0.0059)
ρ_{CP}		0.2825	0.2864
(s.e.)		(0.0089)	(0.0073)
ν			2.9974
(s.e.)			(0.1736)
log-likelihood	-8,291.897	-8,188.390	-8,235.523
numb. of parms.	1	3	4
AIC	16,585.79	16,382.78	16,479.05

Approaches to aggregating risks

- Standard methodology - based on the following assumptions:
  (i) $\mathbf{X} = (X_1, \ldots, X_n)'$ follows a multivariate normal with mean $\mu = (\mu_1, \ldots, \mu_n)'$ and covariance $\mathbf{\Sigma} = (\sigma_{ij})$; and
  (ii) The risk measure used is the quantile risk measure or VaR.
- Extension to the standard methodology - based on the following assumptions:
  (i) Each X_i belongs to a location-scale family of distributions:
  $$X_i = \mu_i + \sigma_i Z, \quad \text{for } i = 1, \ldots, n.$$
  (ii) S also belongs to same location-scale family: $S = \mu_S + \sigma_S Z$; and
  (iii) Risk measure used is conditional tail expectation or TVaR.
- Numerical simulations with copulas.

The standard methodology

S has a normal distribution with mean $\mathbb{E}(S) = \sum_{i=1}^{n} \mu_i$ and variance $\text{Var}(S) = \mathbf{1}'\mathbf{\Sigma}\mathbf{1}$, where $\mathbf{1} = (1, 1, \ldots, 1)'$.

Thus, we have

$$\text{SCR}_S = \text{VaR}_q(S) - \mathbb{E}(S),$$

where, using the property of normal distribution, we have

$$\text{VaR}_q(S) = \Phi^{-1}(q)\sigma_S + \mathbb{E}(S),$$

and hence,

$$\text{SCR}_S = \Phi^{-1}(q)\sigma_S = \Phi^{-1}(q)\sqrt{\text{Var}(S)} = \Phi^{-1}(q)\sqrt{\mathbf{1}'\mathbf{\Sigma}\mathbf{1}}.$$

Φ^{-1} denotes the quantile function of a standard normal and σ_S is the standard deviation of S.

The standard methodology - continued

Note that

$$\mathbf{1}'\mathbf{\Sigma}\mathbf{1} = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathsf{Cov}(X_i, X_j) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}$$

$$= \frac{1}{[\Phi^{-1}(q)]^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathsf{SCR}_i\mathsf{SCR}_j\rho_{ij} = \frac{1}{[\Phi^{-1}(q)]^2}\mathbf{SCR}'\ \Sigma\ \mathbf{SCR},$$

where

$$\mathbf{SCR} = (\mathsf{SCR}_{X_1}, \ldots, \mathsf{SCR}_{X_n})',$$

the vector of stand-alone solvency capitals SCR_{X_i} for each risk i.

This proof has appeared in Dhaene (2005). It immediately follows that

$$\mathsf{SCR}_S = \sqrt{\mathbf{SCR}'\ \Sigma\ \mathbf{SCR}}.$$

The stand-alone capitals can indeed be written as

$$\mathsf{SCR}_{X_i} = \Phi^{-1}(q)\sigma_{X_i} = \Phi^{-1}(q)\sqrt{\mathsf{Var}(X_i)}.$$

Extension to the standard methodology

For stand-alone losses X_i, we have

$$\begin{aligned}\mathrm{TVaR}_q(X_i) &= \mathbb{E}(X_i|X_i > \mathrm{VaR}_q(X_i)) \\ &= \mu_i + \sigma_i \mathbb{E}(Z|Z > \mathrm{VaR}_q(Z)) \\ &= \mu_i + \sigma_i \mathrm{TVaR}_q(Z).\end{aligned}$$

Similarly, we have $\mathrm{TVaR}_q(S) = \mu_S + \sigma_S \mathrm{TVaR}_q(Z)$.

From here, we find that

$$\begin{aligned}\mathbf{1}'\mathbf{\Sigma}\mathbf{1} &= \frac{1}{[\mathrm{TVaR}_q(Z)]^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathrm{TVaR}_q(X_i)-\mu_i)\rho_{ij}(\mathrm{TVaR}_q(X_j)-\mu_j) \\ &= \frac{1}{[\mathrm{TVaR}_q(Z)]^2}(\mathrm{TVaR}_q(\mathbf{X})-\mu)'\mathbf{\Sigma}(\mathrm{TVaR}_q(\mathbf{X})-\mu).\end{aligned}$$

where $\mathrm{TVaR}_q(\mathbf{X}) = (\mathrm{TVaR}_q(\mathbf{X_1}), \ldots, \mathrm{TVaR}_q(\mathbf{X_n}))'$, the vector of stand-alone solvency capitals $\mathrm{TVaR}_q(\mathbf{X_i})$ for each risk i.

Extension - continued

It follows that

$$\mathrm{SCR}_S = \mu_S + \sqrt{(\mathrm{TVaR}_q(\mathbf{X}) - \mu)' \, \mathbf{\Sigma} \, (\mathrm{TVaR}_q(\mathbf{X}) - \mu)}.$$

A similar form to the standard methodology can be found in this case:

$$\mathrm{SCR}_S = \mu_S + \sqrt{\mathbf{SCR}' \, \mathbf{\Sigma} \, \mathbf{SCR}}.$$

Indeed, Dhaene (2005) provides a further extension to the class of distortion risk measures for which the Tail VaR is a special case.

This class of risk measures was introduced by Wang (1996).

Some useful references

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