Principles and Methods of Capital Allocation for Enterprise Risk Management

Lecture 1 of 4-part series

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Outline

The allocation of capital

Capital allocation is the term typically referring to the subdivision of a company's aggregate capital across its various constituents:

- lines of business
- its subsidiaries
- product types within lines of business
- territories, e.g. distribution channels
- types of risks: e.g. market, credit, pricing/underwriting, operational

Company is typically involved in the financial services industry e.g. banks, insurance companies.

A very important component of *Enterprise Risk Management*:

- identifying, measuring, pricing and controlling risks

The purpose of capital

Knowing how much capital you need for your overall business is a key aspect of ERM.

Capital is the amount set aside, usually in excess of assets backing all liabilities, so that the firm:

- could withstand and absorb “unexpected losses” from all risks it is facing;
- would remain solvent with high probability; and
- is able to cover obligations to its customers as promised.

Economic capital vs regulatory capital:

- Economic capital is usually calculated based on true market value (or economic) terms.
- Regulatory capital is usually calculated on the basis of prescribed guidelines by regulatory authorities.

Risk measures for capital computations

We will assume that how we measure capital is known and given.

- Requires understanding all aspects of risks (or losses) the company is facing.
  - modeling the distribution of losses
  - understanding expectation and variation of these losses
  - understanding possible inter-dependencies of these losses
- Some well known risk measures may be used:
  - Value-at-Risk or percentile or VaR
  - Conditional tail expectation or Tail-VaR
  - If X is the random loss, then $\rho[X]$ is some risk measure.

Risk measures - quick review

A risk measure is a mapping ρ from a set Γ of real-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$:

$$\rho : \Gamma \to \mathbb{R} : X \in \Gamma \to \rho[X].$$

Let $X, X_1, X_2 \in \Gamma$. Some well known properties that risk measures may or may not satisfy:

- *Law invariance*: If $\mathbb{P}[X_1 \leq x] = \mathbb{P}[X_2 \leq x]$ for all $x \in \mathbb{R}$, $\rho[X_1] = \rho[X_2]$.
- *Monotonicity*: $X_1 \leq X_2$ implies $\rho[X_1] \leq \rho[X_2]$.
- *Positive homogeneity*: For any $a > 0$, $\rho[aX] = a\rho[X]$.
- *Translation invariance*: For $b \in \mathbb{R}$, $\rho[X + b] = \rho[X] + b$.
- *Subadditivity*: $\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2]$.

Some important concepts

Conditional Tail Expectation (CTE): (sometimes called TailVaR)

$$\mathrm{CTE}_p[X] = \mathbb{E}\left[X|X > F_X^{-1}(p)\right], \qquad p \in (0,1).$$

In general, not subadditive, but it is so for continuous random variables.

Comonotonic sum: $S^c = \sum_{i=1}^n F_{X_i}^{-1}(U)$ where U is uniform on $(0,1)$.

The Fréchet bounds:

$$L_F(u_1,\ldots,u_n) \le C(u_1,\ldots,u_n) \le U_F(u_1,\ldots,u_n),$$

where

Fréchet lower bound: $L_F = \max\left(\sum_{i=1}^n u_i - (n-1), 0\right)$, and
Fréchet upper bound: $U_F = \min(u_1,\ldots,u_n)$.

Some special distributions

Distribution	density $f_X(x)$	Quantile $Q_p[X]$	$\mathrm{CTE}_p[X]$
Normal	$\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\mu + \Phi^{-1}(p)\sigma$	$\mu + \frac{\phi(\Phi^{-1}(p))}{p}\sigma$
Gamma	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\beta x}$	no explicit form	$\frac{\overline{F}_X\left(x_p;\alpha+1,\beta\right)}{\overline{F}_X\left(x_p;\alpha,\beta\right)}\frac{\alpha}{\beta}$
Lognormal	$\frac{1}{\sqrt{2\pi}\sigma x}\mathrm{e}^{-\frac{1}{2}\left(\frac{\log(x)-\mu}{\sigma}\right)^2}$	$\mathrm{e}^{\mu+\Phi^{-1}(p)\sigma}$	$\mathrm{e}^{\mu+\sigma^2/2}\frac{\Phi\left(\sigma-\Phi^{-1}(p)\right)}{1-p}$
Pareto	$\frac{ab^a}{(x+b)^{a+1}}$	$b\left[(1-p)^{-1/a}-1\right]$	$\frac{a}{a-1}Q_p[X]+\frac{b}{a-1}$

Illustrative case study

For purposes of showing illustrations, we will consider an insurance company with five lines of business:

- auto insurance - property damage
- auto insurance - liability
- household or homeowners' insurance
- professional liability
- other lines of business

We will measure loss on a per premium basis and denote the random variable by S for the entire company and X_i for the i-th line of business, $i = 1, 2, 3, 4, 5$.

Model assumptions are described in the subsequent slides.

Model assumptions

Line of business	Loss distribution	Premium share	Parameters	Mean	Variance
Auto (PD)	Gamma	30%	$\alpha = 360$, $\beta = 600$	0.60	0.001
Auto (liab)	Lognormal	20%	$\mu = -0.362$, $\sigma = 0.101$	0.70	0.005
Household	Gamma	15%	$\alpha = 56.25$, $\beta = 75.0$	0.75	0.01
Prof liab	Pareto	15%	$a = 6.92$, $b = 4.74$	0.80	0.90
Other	Lognormal	20%	$\mu = -0.784$, $\sigma = 0.427$	0.50	0.05

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0.10 & 0.10 & 1.00 & & \\
0.20 & 0.50 & 0.10 & 1.00 & \\
0.05 & 0.20 & 0.10 & 0.40 & 1.00
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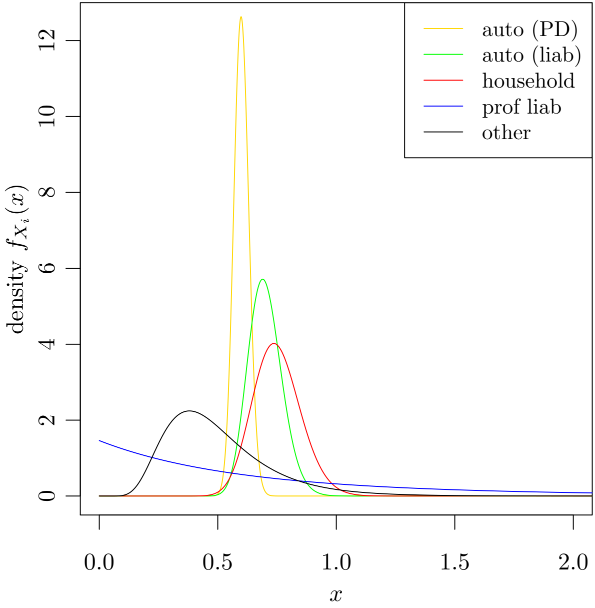
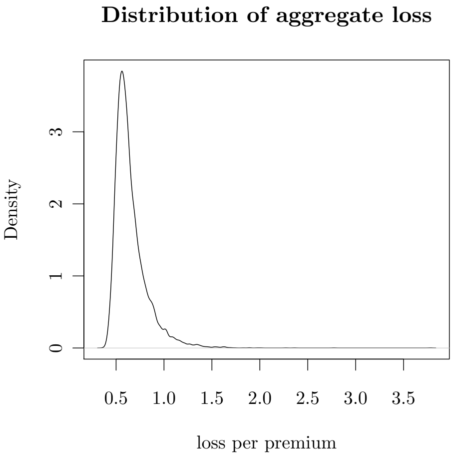
correlation between lines of business

Graph of densities - by lines of business



Distribution of the aggregate loss

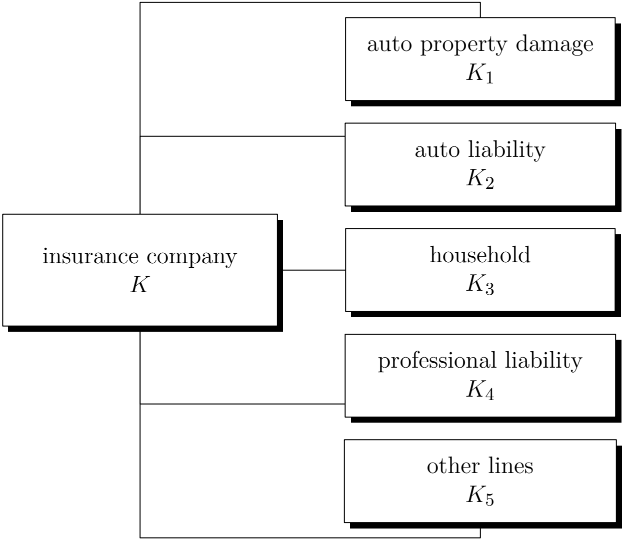


mean	SD	median	min	max	$\mathrm{VaR}_{0.95}[S]$	$\mathrm{CTE}_{0.95}[S]$
0.6536	0.1746	0.6085	0.3611	3.7820	0.9831	1.1758

Stand-alone capitals

Line of business	$\mathrm{VaR}_{0.95}[X_i]$	$\mathrm{CTE}_{0.95}[X_i]$
Auto (PD)	0.6532	0.6679
Auto (liab)	0.8226	0.8586
Household	0.9223	0.9722
Prof liab	2.6139	3.7320
Other	0.9384	1.1286

Insurance company with multiple lines of business



Proportional capital allocation

Many well-known allocation formulas fall into a class of proportional allocations.

Members of this class are obtained by first choosing a risk measure ρ and then attributing the capital $K_i = \gamma_i \rho\left[X_i\right]$ to each business unit i, $i = 1, \ldots, n$.

The factor γ_i is chosen such that the full allocation requirement is satisfied.

This gives rise to the proportional allocation principle:

$$K_i = \frac{K}{\sum_{j=1}^{n} \rho[X_j]} \rho[X_i], \qquad i = 1, \ldots, n.$$

Covariance capital allocation

Because of its popularity, we also consider here for purposes of early illustrations this allocation using covariance.

The covariance is based on the fact that when we have an aggregate loss that is a weighted sum such as

$$S = \sum_{j=1}^{n} c_j X_j,$$

then it is easy to see that

$$\mathrm{Var}[S] = \mathrm{Cov}\left[\sum_{j=1}^{n} c_j X_j, S\right] = \sum_{j=1}^{n} c_j \mathrm{Cov}[X_j, S]$$

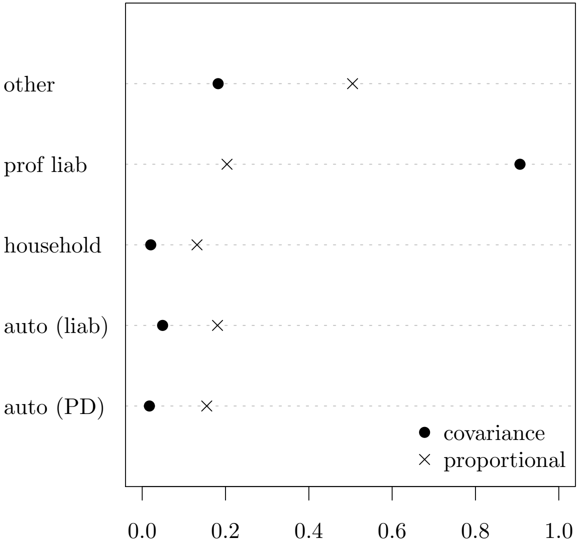
In some sense, this is a special case of the proportional allocation formula with the factor γ_i chosen that gives rise to the covariance allocation principle:

$$K_i = \frac{c_i \mathrm{Cov}[X_i, S]}{\mathrm{Var}[S]} K, \qquad i = 1, \ldots, n.$$

Proportional and covariance allocation results

	proportional allocation based on		covariance allocation based on	
Line of business	VaR	CTE	VaR	CTE
Auto (PD)	0.1786	0.1808	0.0144	0.0173
Auto (liab)	0.1500	0.1549	0.0409	0.0489
Household	0.1261	0.1316	0.0171	0.0205
Prof liab	0.3574	0.5050	0.7583	0.9069
Other	0.1711	0.2036	0.1524	0.1823
Total	0.9831	1.1758	0.9831	1.1758

Results of covariance vs proportional allocations



Selected reference

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