Principles and Methods of Capital Allocation for Enterprise Risk Management

Lecture 2 of 4-part series

Spring School on Risk Management, Insurance and Finance
European University at St. Petersburg, Russia

2-4 April 2012

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Outline

Alternative definitions of economic capital

The Specialty Guide on Economic Capital, page 7, prepared by the SOA Risk Management Task Force (RMTF) headed by Hubert Mueller in 2003, offers the following alternative definitions of economic capital:

- Definition #1

  Economic capital is defined as sufficient surplus to meet potential negative cash flows and reductions in value of assets or increases in value of liabilities at a given level of risk tolerance, over a specified time horizon.

- Definition #2

  Economic capital is defined as the excess of the market value of the assets over the fair value of liabilities required to ensure that obligations can be satisfied at a given level of risk tolerance, over a specified time horizon.

- Definition #3

  Economic capital is defined as sufficient surplus to maintain solvency at a given level of risk tolerance, over a specified time horizon.

The purpose of economic capital

Calculating the economic capital for a firm has its many purposes (this list not all encompassing):

- understand the company's overall risk profile
- capital budgeting
- pricing purposes
- company asset and liability management
- understanding company's risk tolerances and possible constraints
- performance measurement and incentive compensation
- regulatory and other external parties (e.g. investors, ratings agencies)

Many of these can be found in the Specialty Guide; a lot of development since its publication.

Capital allocation and product pricing

For simplicity, consider an insurer faced with pricing a risk Z, only for a given short horizon, and assume the company has risk tolerance described by an exponential utility function:

$$u(x) = \frac{1}{a}\left(1 - \mathrm{e}^{-ax}\right), \quad \text{for } a > 0.$$

It is rather straightforward to show that the smallest amount of premium this company is willing to accept in exchange of facing the risk can be expressed as:

$$P = \frac{1}{a} \log \mathbb{E}\left[\mathrm{e}^{aZ}\right]$$

Now, suppose the company is faced with the task of pricing a portfolio of n of these identical risks, say $X_1, X_2, \ldots, X_n$ so that the aggregate risk is

$$S = X_1 + X_2 + \cdots + X_n$$

- continued

The total (minimum) premium the company needs to collect is

$$P = \frac{1}{a} \log \mathbb{E}\left[e^{a \sum_{j=1}^{n} X_j} \right],$$

for which we can express as, assuming independent risks:

$$P = \frac{1}{a} \log \prod_{j=1}^{n} \mathbb{E}\left[e^{aX_j} \right] = \sum_{j=1}^{n} \frac{1}{a} \log \mathbb{E}\left[e^{aX_j} \right].$$

Each risk then can be assessed a premium of

$$P_j = \frac{1}{a} \log \mathbb{E}\left[e^{aX_j} \right], \quad \text{for } j = 1, 2, \ldots, n$$

However, if we relax the assumption of independence:

$$P = \frac{1}{a} \log \prod_{j=1}^{n} \mathbb{E}\left[e^{aX_j} \right] \leq \sum_{j=1}^{n} \frac{1}{a} \log \mathbb{E}\left[e^{aX_j} \right] = P_1 + P_2 + \cdots + P_n$$

Some effect of diversification!!!

The allocation problem

- Consider a portfolio of n individual losses $X_1, \ldots, X_n$ during some well-defined reference period.
- Assume these random losses have a dependency structure characterized by the joint distribution of the random vector $(X_1, \ldots, X_n)$.
- The aggregate loss is the sum $S = \sum_{i=1}^{n} X_i$, or in some instances, it could be a weighted sum $S = \sum_{i=1}^{n} w_i X_i$.
- Assume company holds aggregate level of capital K which may be determined from a risk measure ρ such that $K = \rho(S) \in \mathbb{R}$.
- Here the capital (economic) is the smallest amount the company must set aside to withstand aggregated losses at an acceptable level, and tolerance.

- continued

- The company now wishes to allocate K across its various business units.
    - determine non-negative real numbers $K_1, \ldots, K_n$ satisfying:

$$\sum_{i=1}^{n} K_i = K.$$

- This requirement is referred to as "the full allocation" requirement.
- We will see that this requirement is a constraint in our optimization problem (in the optimal paper).

The literature

There are countless number of ways to allocate aggregate capital.

Good overview of methods:

- Cummins (2000); Venter (2004)

Some methods based on decision making tools:

- Cummins (2000)- RAROC, EVA
- Lemaire (1984); Denault (2001) - game theory
- Tasche (2004) - marginal costs
- Kim and Hardy (2008) - solvency exchange option with limited liability

This list needs some updating.

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Some methods based on risk measures/distributions:

- Panjer (2001) - TVaR, multivariate normal
- Landsman and Valdez (2003) - TVaR, multivariate elliptical
- Dhaene, et al. (2008) - TVaR, lognormal
- Valdez and Chernih (2003) - covariance-based allocation, multivariate elliptical
- Tsanakas (2004, 2008) - distortion risk measures, convex risk measures
- Furman and Zitikis (2008) - weighted risk capital allocations

Methods also based on optimization principle:

- Dhaene, Goovaerts and Kaas (2003); Laeven and Goovaerts (2004); Zaks, Frostig and Levikson (2006)

Some notation

Denote by $\mathbf{X}^T = (X_1, X_2, \ldots, X_n)$ the vector of losses, with each entry denoting the loss for the applicable line of business.

Define an allocation A to be a mapping $A : \mathbf{X}^T \to \mathbb{R}^n$ such that

$$A(\mathbf{X}^T) = (K_1, K_2, \ldots, K_n),$$

where the “full allocation” is satisfied:

$$K = \rho[Z] = \sum_{i=1}^{n} K_i$$

Each component K_i of the allocation is viewed as the i-th line of business contribution to the total company capital.

Because allocation must also reflect the fact that each line operates in the presence of the other lines, the notation

$$K_i = A(X_i | X_1, \ldots, X_n)$$

may be well suited for this purpose.

Possible criteria for a fair allocation

Let $N = \{1, 2, \ldots, n\}$ be the set of the first n positive integers.

- **No undercut**: For any subset $M \subseteq N$, we have

$$\sum_{i \in M} A(X_i | X_1, \ldots, X_n) \leq \rho\left[\sum_{i \in M} X_i\right]$$

- **Symmetry**: Let $N^* = N - \{i_1, i_2\}$. If $M \subset N^*$ (strict subset) with $|M| = m$, $\mathbf{X}_m^T = \left(X_{j_1}, \ldots, X_{j_m}\right)$ and if

$$A\left(X_{i_1} | \mathbf{X}_m^T, X_{i_1}, X_{i_2}\right) = A\left(X_{i_2} | \mathbf{X}_m^T, X_{i_1}, X_{i_2}\right)$$

  for every $M \subset N^*$, then we must have $K_{i_1} = K_{i_2}$.

- **Consistency**: For any subset $M \subseteq N$ with $|M| = m$, let $\mathbf{X}_{n-m}^T = \left(X_{j_1}, \ldots, X_{j_{n-m}}\right)$ for all $j_k \in N - M$ where $k = 1, 2, \ldots, n - m$. Then we have

$$\sum_{i \in M} A(X_i | X_1, \ldots, X_n) = A\left(\sum_{i \in M} X_i \,\middle|\, \sum_{i \in M} X_i, \mathbf{X}_{n-m}^T\right).$$

Intuitive interpretations

The no undercut criterion:

- recognizes the benefits of diversification
- risk allocated to a business unit may not exceed the risk allocated if these were offered on a stand-alone basis
- analogous to sub-additivity for a coherent risk measure

The symmetry criterion:

- states that two lines of business that equally contribute to the risk within the firm must have equal allocation
- allocation must therefore recognize only the level or degree of contribution to risk within the firm, and nothing else
- sometimes called "equitability" property

The consistency criterion:

- ensures that a unit's allocation cannot depend on the level at which the allocation occurs
- allocation is independent of the hierarchical structure of the firm

Relative or proportional allocation method

Although quite a popular method, partly because of its simplicity, the relative allocation formula according to

$$A(X_i|X_1,\ldots,X_n) = \frac{\rho[X_i]}{\rho[X_1]+\cdots+\rho[X_n]} \times \rho[S]$$

violates all three possible criteria for a fair allocation.

See Valdez and Chernih (2003) for proofs.

The covariance allocation method

The covariance allocation formula is based on

$$A(X_i|X_1,\ldots,X_n) = \lambda_i \times \rho[S],$$

$\lambda^T = (\lambda_1,\ldots,\lambda_n)$ denotes a vector of weights that adds up to one so that full allocation is satisfied.

To determine these weights λ_i, one approach is to minimize the following quadratic loss function:

$$\mathbb{E}\left[\left((\mathbf{X}-\mu)-\lambda(S-\mu_S)\right)^T \mathbf{W}\left((\mathbf{X}-\mu)-\lambda(S-\mu_S)\right)\right]$$

where the weight-matrix $\mathbf{W}$ is assumed to be positive definite.

Optimization results to the following unique values of:

$$\lambda_i = \frac{\mathbb{E}\left[(X_i-\mu_i)(S-\mu_S)\right]}{\mathbb{E}\left[(S-\mu_S)^2\right]} = \frac{\text{Cov}[X_i,S]}{\text{Var}[S]},$$

for $i = 1,\ldots,n$.

- continued

The capital allocation formula based on the covariance principle satisfies the three criterion or properties of a possible fair allocation:

- no undercut,
- symmetry, and
- consistency.

See Valdez and Chernih (2003) for proofs.

Wang's capital decomposition formula

Preserving the notation used by Wang (2002), define and denote the expectation of $X_{i,Q}$ by

$$H_\lambda[X_i, S] = \mathbb{E}[X_{i,Q}] = \frac{\mathbb{E}[X \cdot \exp(\lambda S)]}{\mathbb{E}[\exp(\lambda S]}$$

and the expectation of the aggregate loss Z_Q by

$$H_\lambda[S, S] = \mathbb{E}[S_Q] = \frac{\mathbb{E}[S \cdot \exp(\lambda S)]}{\mathbb{E}[\exp(\lambda S]}.$$

Note that this last equation exactly gives the Esscher Transform of S.

The price of a random payment X_i traded in the market is $H_\lambda[X_i, S]$ so that one can think of the difference

$$\rho[X_i] = \mathbb{E}[X_{i,Q}] - \mathbb{E}[X_i] = H_\lambda[X_i, S] - \mathbb{E}[X_i]$$

as the risk premium.

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For the aggregate loss S, its risk premium is given by

$$\rho[S] = \rho\left[\sum_{i=1}^{n} X_i\right] = H_\lambda[S, S] - \mathbb{E}[S].$$

It is rather straightforward to show that

$$\rho[X_i] = \frac{\mathrm{Cov}[X_i, \exp(\lambda S]}{\mathbb{E}[\exp(\lambda S)]} \quad \text{and} \quad \rho[S] = \frac{\mathrm{Cov}[S, \exp(\lambda S]}{\mathbb{E}[\exp(\lambda S)]}.$$

Wang (2002) proposed computing the allocation of capital to individual business unit i based on the following formula:

$$K_i = H_\lambda[X_i, S] - \mathbb{E}[X_i].$$

Assuming an aggregate capital of K, the parameter λ can be computed by solving then

$$K = H_\lambda[S, S] - \mathbb{E}[S].$$

It is not difficult to show that the full allocation requirement is met in this case: $K = \sum_{i=1}^{n} K_i$.

Special case: multivariate normal

In the special case where the vector of losses $(X_1, \ldots, X_n)$ follows a multivariate normal distribution, we have that the Wang's allocation method reduces to the covariance method.

Some straightforward calculation yields the results:

$$\mathbb{E}[S\exp(\lambda S)] = \exp\left(\lambda\mu_S + \frac{1}{2}\lambda^2\sigma_S^2\right) \cdot (\mu + \lambda\sigma_S^2)$$

and

$$\mathbb{E}[X_i\exp(\lambda S)] = \exp\left(\lambda\mu_S + \frac{1}{2}\lambda^2\sigma_S^2\right) \cdot (\mu_i + \lambda\sigma_{i,S})$$

Then it follows that

$$K = \lambda\sigma_S^2 \text{ and } K_i = \lambda\sigma_{i,S}$$

which clearly is equivalent to the covariance method.

Selected reference

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