Principles and Methods of Capital Allocation for Enterprise Risk Management

Lecture 3 of 4-part series

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Outline

Main reference

Dhaene, J., Tsanakas, A., Valdez, E.A., and S. Vanduffel (2012). Optimal capital allocation principles. *Journal of Risk and Insurance*, 79(1), 1-28.

Our contribution to the literature

- We re-formulate the problem as minimum distance problem in the sense that the weighted sum of measure for the deviations of the business unit's losses from their respective capitals be minimized:
  - essentially distances between K_i and X_i
- Takes then into account some important decision making allocation criteria such as:
  - the purpose of the allocation allowing the risk manager to meet specific target objectives
  - the manner in which the various segments interact, e.g. legal and/or organizational structure
- Solution to minimizing distance formula leads to several existing allocation methods. New allocation formulas also emerge.

Some important concepts already discussed

- Capital allocation
  - why important?
- Risk measures
  - economic capital calculation
  - VaR and CTE or Tail-VaR
- Some popular allocation formulas

α–mixed inverse distribution function

For $p \in (0,1)$, we denote the Value-at-Risk (VaR) or quantile of X by $F_X^{-1}(p)$ defined by:

$$F_X^{-1}(p) = \inf \{x \in \mathbb{R} | F_X(x) \geq p\}.$$

We define the inverse distribution function $F_X^{-1+}(p)$ of X as

$$F_X^{-1+}(p) = \sup \{x \in \mathbb{R} | F_X(x) \leq p\}.$$

The α–mixed inverse distribution function $F_X^{-1(\alpha)}$ of X is:

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p).$$

It follows for any X and for all x with $0 < F_X(x) < 1$, there exists an $\alpha_x \in [0,1]$ such that $F_X^{-1(\alpha_x)}(F_X(x)) = x$.

Some familiar allocation methods

Allocation method	$\rho[X_i]$	K_i
Haircut allocation (no known reference)	$F_{X_i}^{-1}(p)$	$\frac{K}{\sum_{j=1}^{n} F_{X_j}^{-1}(p)} F_{X_i}^{-1}(p)$
Quantile allocation Dhaene et al. (2002)	$F_{X_i}^{-1(\alpha)}\left(F_{S^c}(K)\right)$	$F_{X_i}^{-1(\alpha)}\left(F_{S^c}(K)\right)$
Covariance allocation Overbeck (2000)	$\text{Cov}[X_i, S]$	$\frac{K}{\text{Var}[S]} \text{Cov}[X_i, S]$
CTE allocation Acerbi and Tasche (2002), Dhaene et al. (2006)	$\mathbb{E}\left[X_i \middle\vert S > F_S^{-1}(p)\right]$	$\frac{K}{\text{CTE}_p[S]} \mathbb{E}\left[X_i \middle\vert S > F_S^{-1}(p)\right]$

The optimal capital allocation problem

We reformulate the allocation problem in terms of optimization:

Given the aggregate capital $K > 0$, we determine the allocated capitals $K_i,\ i = 1, \ldots, n$, from the following optimization problem:

$$\min_{K_1,\ldots,K_n} \sum_{j=1}^{n} v_j \, \mathbb{E}\left[\zeta_j \; D\left(\frac{X_j - K_j}{v_j}\right)\right]$$

such that the full allocation is met:

$$\sum_{j=1}^{n} K_j = K,$$

and where the v_j's are non-negative real numbers such that $\sum_{j=1}^{n} v_j = 1$, the ζ_j are non-negative random variables such that $\mathbb{E}[\zeta_j] = 1$ and D is a non-negative function.

The components of the optimization

Elaborating on the various elements of the optimization problem:

- Distance measure: the function $D(\cdot)$ gives the deviations of the outcomes of the losses X_j from their allocated capitals K_j.
  - squared-error or quadratic: $D(x) = x^2$
  - absolute deviation: $D(x) = |x|$
- Weights: the random variable ζ_j provides a re-weighting of the different possible outcomes of these deviations.
- Exposure: the non-negative real number v_j measures exposure of each business unit according to for example, revenue, premiums, etc.

The case of the quadratic optimization

Consider the special case of quadratic optimization where

$$D(x) = x^2$$

so that the optimization is expressed as

$$\min_{K_1,\ldots,K_n} \sum_{j=1}^{n} \mathbb{E}\left[\zeta_j \frac{(X_j - K_j)^2}{v_j}\right].$$

This optimal allocation problem has the following unique solution:

$$K_i = \mathbb{E}[\zeta_i X_i] + v_i \left(K - \sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j]\right), \qquad i = 1, \ldots, n.$$

Business unit driven allocations

Risk measure	$\zeta_i = h_i(X_i)$	$\mathbb{E}[X_i h_i(X_i)]$
Standard deviation principle Buhlmann (1970)	$1 + a\dfrac{X_i - \mathbb{E}[X_i]}{\sigma_{X_i}},\ a \geq 0$	$\mathbb{E}[X_i] + a\sigma_{X_i}$
Conditional tail expectation Overbeck (2000)	$\dfrac{1}{1-p}\mathbb{I}\left(X_i > F_{X_i}^{-1}(p)\right),\ p \in (0,1)$	$\text{CTE}_p\left[X_i\right]$
Distortion risk measure Wang (1996), Acerbi (2002)	$g'\left(\overline{F}_{X_i}(X_i)\right), g: [0,1] \mapsto [0,1],$ $g' > 0, g'' < 0$	$\mathbb{E}\left[X_i g'\left(\overline{F}_{X_i}(X_i)\right)\right]$
Exponential principle Gerber (1974)	$\displaystyle\int_0^1 \frac{e^{\gamma a X_i}}{\mathbb{E}[e^{\gamma a X_i}]} d\gamma,\ a > 0$	$\dfrac{1}{a}\ln \mathbb{E}\left[e^{aX_i}\right]$
Esscher principle Gerber (1981)	$\dfrac{e^{aX_i}}{\mathbb{E}[e^{aX_i}]},\ a > 0$	$\dfrac{\mathbb{E}[X_i e^{aX_i}]}{\mathbb{E}[e^{aX_i}]}$

Aggregate portfolio driven allocations

Reference	$\zeta_i = h(S)$	$\mathbb{E}[X_i h(S)]$
Overbeck (2000)	$1 + a\dfrac{S - \mathbb{E}[S]}{\sigma_S},\ a \geq 0$	$\mathbb{E}[X_i] + a\dfrac{\text{Cov}[X_i, S]}{\sigma_S}$
Overbeck (2000)	$\dfrac{1}{1-p}\mathbb{I}\left(S > F_S^{-1}(p)\right),\ p \in (0,1)$	$\mathbb{E}[X_i \| S > F_S^{-1}(p)]$
Tsanakas (2004)	$g'(\overline{F}_S(S)), g : [0,1] \mapsto [0,1], g' > 0,$ $g'' < 0$	$\mathbb{E}\left[X_i g'(\overline{F}_S(S))\right]$
Tsanakas (2008)	$\int_0^1 \dfrac{e^{\gamma a S}}{\mathbb{E}[e^{\gamma a S}]} d\gamma,\ a > 0$	$\mathbb{E}\left[X_i \int_0^1 \dfrac{e^{\gamma a S}}{E[e^{\gamma a S}]} d\gamma\right]$
Wang (2007)	$\dfrac{e^{aS}}{\mathbb{E}[e^{aS}]},\ a > 0$	$\dfrac{\mathbb{E}[X_i e^{aS}]}{\mathbb{E}[e^{aS}]}$

Market driven allocations

Let ζ_M be such that market-consistent values of the aggregate portfolio loss S and the business unit losses X_i are given by $\pi[S] = \mathbb{E}[\zeta_M S]$ and $\pi[X_i] = \mathbb{E}[\zeta_M X_i]$.

To determine an optimal allocation over the different business units, we let $\zeta_i = \zeta_M,\ i = 1, \ldots, n$, allowing the market to determine which states-of-the-world are to be regarded adverse. This yields:

$$K_i = \pi[X_i] + v_i \left(K - \pi[S] \right).$$

Using market-consistent prices as volume measures $v_i = \pi[X_i]/\pi[S]$, we find

$$K_i = \frac{K}{\pi[S]} \pi[X_i], \qquad i = 1, \ldots, n.$$

Rearranging these expressions leads to

$$\frac{K_i - \pi[X_i]}{\pi[X_i]} = \frac{K - \pi[S]}{\pi[S]}, \qquad i = 1, \ldots, n.$$

Allocation with respect to the default option

An alternative choice for the weighting random variable is

$$\zeta_i = \frac{\mathbb{I}(S > K)}{\mathbb{P}[S > K]}, \qquad i = 1, \ldots, n,$$

such that only those states-of-the-world that correspond to insolvency are considered.

The allocation rule then becomes

$$K_i = \mathbb{E}\left[X_i | S > K\right] + v_i \left(K - \mathbb{E}\left[S | S > K\right]\right).$$

which can be rearranged as follows:

$$\mathbb{E}[(X_i - K_i)\, \mathbb{I}(S > K)] = v_i\, \mathbb{E}[(S - K)_+], \qquad i = 1, \ldots, n.$$

Quantity $\mathbb{E}[(S - K)_+]$ represents the expected policyholder deficit.

Marginal contribution of each business unit to the EV of the policyholder deficit is the same per unit of volume, and hence consistent with Myers and Read Jr. (2001).

Additional items considered in the paper

- We also considered other optimization criterion:
  - absolute value deviation: $D(x) = |x|$
  - combined quadratic and shortfall: $D(x) = ((x)_+))^2$
  - shortfall: $D(x) = (x)_+$
- Shortfall is applicable in cases where insurance market guarantees payments out of a pooled fund contributed by all companies, e.g. Lloyd's.
- Such allocation can be posed as an optimization problem leading to formulas that have been considered by Lloyd's.

[Note: views here are the authors' own and do not necessarily reflect those of Lloyd's.]

Illustrative case study - same as previous

For purposes of showing illustrations, we consider an insurance company with five lines of business:

- auto insurance - property damage
- auto insurance - liability
- household or homeowners' insurance
- professional liability
- other lines of business

We measure loss on a per premium basis and denote the random variable by S for the entire company and X_i for the i-th line of business, $i = 1, 2, 3, 4, 5$.

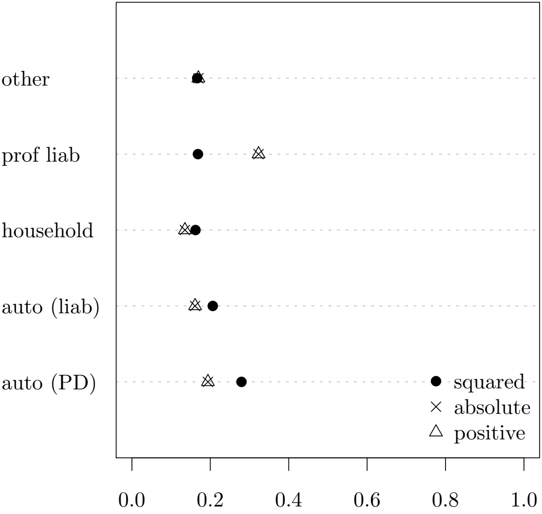
Same model assumption as previously explored.

For convenience and for illustrative purpose only, we set $\zeta_i = 1$ so that risk aversion is ignored.

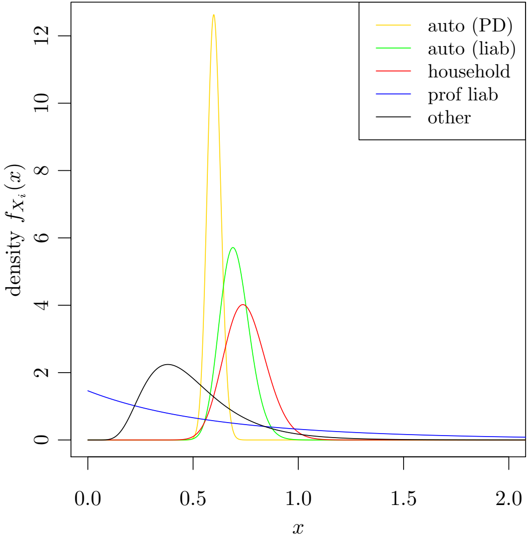
Results of allocation for different distance measure

	allocation based on the distance measure		
Line of business	squared x^2	absolute $\|x\|$	positive $(x)_+$
Auto (PD)	0.2795192	0.1939404	0.1939598
Auto (liab)	0.2063209	0.1612151	0.1612055
Household	0.1623007	0.1352530	0.1352692
Prof liab	0.1684604	0.3230569	0.3230762
Other	0.1664988	0.1696345	0.1695894
Total	0.9831	0.9831	0.9831

Allocation by optimization for different distance measures



Graph of densities - by lines of business



* reproduced here again for convenience

Concluding remarks

- We re-examine existing allocation formulas that are in use in practice and existing in the literature. We re-express the allocation issue as an optimization problem.
- No single allocation formula may serve multiple purposes, but by expressing the problem as an optimization problem it can serve us more insights.
- Each of the components in the optimization can serve various purposes.
- This allocation methodology can lead to a wide variety of other allocation formulas.