MATH 3630

Actuarial Mathematics I

Class Test 2

Wednesday, 18 November 2009

Time Allowed: 1 hour

Total Marks: 100 points

Please write your name and student number at the spaces provided:

Name: SOLUTIONS **Student ID:** EMIL

- **There are ten (10) written-answer questions here and you are to answer all ten. Each question is worth 10 points.**
- **Please provide details of your workings in the appropriate spaces provided; partial points will be granted.**
- **Please write legibly.**
- **Anyone caught writing after time has expired will be given a mark of zero.**

Question No. 1:

For a whole life insurance of a benefit of 100 on (30) payable at the moment of death, you are given:

$$\mu_{30+t} = 0.01, \quad \text{for all } t > 0$$

and

$$\delta_t = \begin{cases} 0.04, & \text{for } 0 < t \leq 5 \\ 0.05, & \text{for } t > 5 \end{cases}.$$

Calculate the Actuarial Present Value of the benefit for this life insurance.

$$APV(\text{benefit}) = 100 * \left[\int_0^5 e^{-.04t} \, .01 \, e^{-.01t} \, dt + \int_5^\infty e^{-.04(5)} e^{-\int_5^t .05 ds} \, .01 e^{-.01t} \, dt\right]$$

$$= 100 * \left[\frac{.01}{.05}\left(1 - e^{-.05(5)}\right) + \underbrace{e^{-.2} e^{.25} \, .01 \int_5^\infty e^{-.06t} \, dt}_{e^{.05} \frac{.01}{.06} e^{-.06(5)}} \right]$$

$$= 100 * [.04424 + .1298]$$

$$= \underline{\underline{17.4040}}$$

Question No. 2:

Let Z_1 denote the present value random variable of an n-year term insurance of \$1, while Z_2 that of an n-year deferred insurance of \$1, with death benefit payable at the moment of death of (x).

You are given:

$$\bar{A}^{\,1}_{x:\overline{n}|} = 0.01 \qquad {}^2\bar{A}^{\,1}_{x:\overline{n}|} = 0.0005$$

and

$$_{n|}\bar{A}_x = 0.10 \qquad {}^2_{n|}\bar{A}_x = 0.0136$$

Calculate the correlation coefficient of Z_1 and Z_2.

In case you forgot: $\mathrm{Corr}(Z_1, Z_2) = \dfrac{\mathrm{Cov}(Z_1, Z_2)}{\sqrt{\mathrm{Var}(Z_1)\mathrm{Var}(Z_2)}}$.

$$Z_1 = \begin{cases} v^T, & T \le n \\ 0, & T > n \end{cases} \quad \text{and} \quad Z_2 = \begin{cases} 0, & T \le n \\ v^T, & T > n \end{cases} \quad \text{clearly } Z_1 Z_2 = 0$$

Thus, $\mathrm{Cov}(Z_1, Z_2) = \cancel{E(Z_1 Z_2)}^{0} - E(Z_1)E(Z_2) = -\bar{A}^{\,1}_{x:\overline{n}|} * {}_{n|}\bar{A}_x$

$= .01 * .1 = \underline{.001}$

$$\mathrm{Var}(Z_1) = {}^2\bar{A}^{\,1}_{x:\overline{n}|} - \left(\bar{A}^{\,1}_{x:\overline{n}|}\right)^2 = .0005 - (.01)^2 = .0004$$

$$\mathrm{Var}(Z_2) = {}^2_{n|}\bar{A}_x - \left({}_{n|}\bar{A}_x\right)^2 = .0136 - (.10)^2 = \underline{.0036}$$

$$\therefore \mathrm{CORR}(Z_1, Z_2) = \frac{-.001}{\sqrt{(.0004)(.0036)}} = \frac{-.001}{(.02)(.06)} = \underline{-.8333}$$

OR

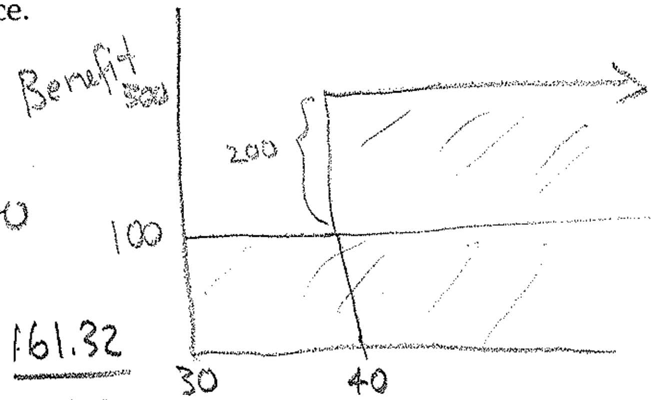
$\underline{-5/6}$

Question No. 3:

For a special life insurance on (30), you are given:

- mortality follows the *Illustrative Life Table* with $i = 6\%$;
- death benefit is payable at the end of the year of death; and
- death benefit is 100 for the first 10 years, and 300 thereafter.

Calculate the Actuarial Present Value of this insurance.



$$APV = 100\, A_{30} + {}_{10}E_{30} * 200\, A_{40}$$

$$= 100\, \frac{102.48}{1000} + \frac{547.33}{1000} * 200\, \frac{161.32}{1000}$$

$$= \underline{\underline{27.90706}}$$

Question No. 4:

An insurance company has a portfolio of 100 whole life insurance policies all issued to age x. You are given:

- The future lifetimes of the policyholders are independent.
- The benefit of each policyholder is \$5 payable at the end of the year of death.
- $A_x = 0.35$
- ${}^2A_x = 0.15$
- $i = 4\%$

Using Normal approximation, calculate the amount of the fund needed at issue in order to be 95% certain of having enough money to pay the death benefits. (Note: the 95$^{\text{th}}$ percentile of a standard Normal is 1.645.)

Let L = total loss $= 5\sum_{j=1}^{100} Z_j$ where $Z_j = v^{K+1}$, $j=1,\dots,100$

$$E(Z_j) = A_x = .35$$

$$Var(Z_j) = {}^2A_x - (A_x)^2 = .15 - (.35)^2 = .0275$$

Thus, $E(L) = 5 * 100 * .35 = 175$

$$Var(L) = 5^2 * 100 * .0275 = 68.75$$

Let F be the required amount of fund so that

$$P(L \le F) = .95 \Rightarrow P\left(Z \le \underbrace{\frac{F-175}{\sqrt{68.75}}}\right) \approx .95$$

$$F = 175 + 1.645\sqrt{68.75} \Leftarrow \frac{F-175}{\sqrt{68.75}} = 1.645$$

$$= 188.6396$$

Question No. 5:

Suppose you are given:

$$_kp_x = (0.95)^k, \quad \text{for } k = 0, 1, \ldots$$

Let Y denote the present value random variable of a whole life annuity-due of \$1 each year issued to (x). Calculate the expected value of Y if $i = 10\%$.

Use current payment technique

$$E(Y) = \ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{1.1}\right)^k (.95)^k$$

$$= \frac{1}{1 - \frac{.95}{1.1}} = \frac{1.10}{.15} = \frac{110}{15} = \frac{22}{3}$$

$$= \underline{7.3333}$$

Question No. 6:

For a group of insured individuals all age x, let Z be the present value random variable of a whole life insurance that pays $1 at the end of the year of death. The group is made up of half male and half female.

You are given:

$$A_x^{\text{M}} = 0.16 \qquad A_x^{\text{F}} = 0.08 \quad \Rightarrow E(Z) = \tfrac{1}{2}(.16 + .08) = .12$$

and

$$\text{Var}(Z|\text{M}) = 0.04 \qquad \text{Var}(Z|\text{F}) = 0.05$$

where 'M' denotes male and 'F' denotes female.

Calculate the (unconditional) variance Var(Z) for an individual randomly selected from the group.

Let G = gender where $P(G=M) = P(G=F) = 1/2$

$$\text{Var}(Z) = \underbrace{E(\text{Var}(Z|G))} + \underbrace{\text{Var}(E(Z|G))}$$

$$= \tfrac{1}{2}(.04 + .05) + \tfrac{1}{2}\left[(.16-.12)^2 + (.08-.12)^2\right]$$

$$= .045 + \cancel{\tfrac{1}{2}}\left[\cancel{2}(.0016)\right]$$

$$= \underline{\underline{.0466}}$$

Question No. 7:

For a whole life annuity-due of \$100 on (x), payable annually, you are given:

- $q_x = 0.01$
- $q_{x+1} = 0.05$
- $i = 0.05$
- $\ddot{a}_{x+1} = 6.951$

Calculate the change in the Actuarial Present Value of this annuity-due if p_{x+1} is increased by 0.02. Did the Actuarial Present Value increase of decrease? Use your intuition to explain why.

Let new $p^*_{x+1} = p_{x+1} + .02$

Recall using recursion, $\ddot{a}_x = 1 + v p_x + v^2 p_x p_{x+1} \ddot{a}_{x+2}$

$$\text{new } \ddot{a}^*_x = 1 + v p_x + v^2 p_x (p_{x+1} + .02) \ddot{a}_{x+2}$$

$$= \ddot{a}_x + .02 v^2 p_x \ddot{a}_{x+2}$$

Thus, change in APV $= \ddot{a}^*_x - \ddot{a}_x = .02 v^2 p_x \ddot{a}_{x+2}$

$$= .02 \left(\frac{1}{1.05}\right)^2 (.99) \ddot{a}_{x+2}$$

where $\ddot{a}_{x+1} = 6.951 = 1 + v p_{x+1} \ddot{a}_{x+2}$

$$= 1 + \frac{1}{1.05} 0.95 \ddot{a}_{x+2} \Rightarrow \ddot{a}_{x+2} = 6.577421$$

$$\therefore \ddot{a}^*_x - \ddot{a}_x = .02 \left(\frac{1}{1.05}\right)^2 (.99)(6.577421) = 0.1181$$

For \$100 of benefit, this change will be 11.81

The APV was expected to increase because of the increase in the probability of surviving a year later, and because everything else remains, this implies increase in future survival and hence the annuity becomes more expensive.

Question No. 8:

You are given:

- T_x has an exponential distribution with mean 20.
- ${}^2\bar{a}_x = 5.75$

Calculate the force of interest δ.

$T_x \sim$ Exponential $\Rightarrow$ constant hazard with

$$\mu = \frac{1}{E(T_x)} = \frac{1}{20} = .05$$

$${}^2\bar{a}_x = \frac{1}{\mu + 2\delta} = \frac{1}{.05 + 2\delta} = 5.75$$

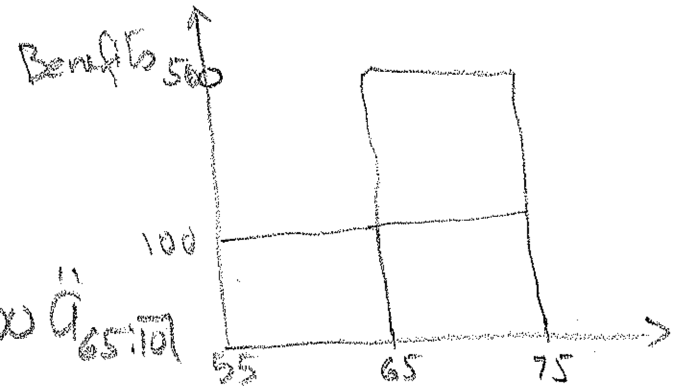
$$\Rightarrow \frac{\frac{1}{5.75} - .05}{2} = \delta = \underline{\underline{.062}}$$

Question No. 9:

For a special 20-year temporary life annuity-due issued to (55), you are given:

- Mortality follows the *Illustrative Life Table* with $i = 6\%$.
- The benefits are \$100 each year for the first 10 years, and \$500 each year the last 10 years.

Calculate the Actuarial Present Value of the benefits for this life annuity.



APV(benefits) =

$$100\,\ddot{a}_{55:\overline{10}|} + {}_{10}E_{55} * 500\,\ddot{a}_{65:\overline{10}|}$$

where $\ddot{a}_{55:\overline{10}|} = \ddot{a}_{55} - {}_{10}E_{55}\,\ddot{a}_{65}$

$$= 12.2758 - \frac{486.96}{1000}(9.8969) = 7.456406$$

$$\ddot{a}_{65:\overline{10}|} = \ddot{a}_{65} - {}_{10}E_{65}\,\ddot{a}_{75}$$

$$= 9.8969 - \left(\frac{399.94}{1000}\right)(7.2170) = 7.010533$$

$$= 100(7.456406) + 500\left(\frac{486.96}{1000}\right)(7.010533)$$

$$= \underline{\underline{2452.565}}$$

Question No. 10:

Raul is currently age 40. His mortality follows DeMoivre's law with $\omega = 100$. He purchases a whole life insurance policy that pays a benefit of \$1 at the moment of death.

Compute the variance of the present value of his death benefit if $\delta = 5\%$.

DeMoivre's law $\Rightarrow T_{40} \sim$ Uniform on $(0, 60)$

$$f_{T_{40}}(t) = \frac{1}{60}, \quad 0 < t \leq 60$$

Let $Z = PV(\text{benefit}) = v^T$

$$\therefore E(Z) = \bar{A}_{40} = \int_0^{60} e^{-.05t} \frac{1}{60}\, dt = \frac{1}{60}\left(\frac{1}{.05}\right)\left(1 - e^{-.05(60)}\right)$$

$$= .3167376$$

$$E(Z^2) = {}^2\bar{A}_{40} = \int_0^{60} e^{-.10t} \frac{1}{60}\, dt = \frac{1}{60}\left(\frac{1}{0.1}\right)\left(1 - e^{-.10(60)}\right)$$

$$= .1662535$$

$$\therefore Var(Z) = E(Z^2) - \left(E(Z)\right)^2$$

$$= .1662535 - (.3167376)^2$$

$$= \underline{\underline{.0659}}$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK