MATH 3630

Actuarial Mathematics I

Final Examination

Monday, 12 December 2008

Time Allowed: 2 hours

Total Marks: 130 points

Please write your name and student number at the spaces provided:

Name: EMIL

Student ID: SUGGESTED SOLUTION

- There are thirteen (13) written-answer questions here and you are to answer all thirteen. Each question is worth 10 points.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught cheating and writing after time has expired will be given a mark of zero.

Question No. 1:

For a fully discrete whole life insurance of $1,000 issued to (x), you are given:

- $q_{x+k} = 0.01$, for all $k \geq 0$; and
- $i = 10\%$.

Calculate the level benefit premium for each year.

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x = \sum_{k=0}^{\infty} \left(\frac{1}{1.1}\right)^k (1-.01)^k = \sum_{k=0}^{\infty} \left(\frac{.99}{1.1}\right)^k$$

$$= \frac{1}{1-\frac{.99}{1.1}} = \frac{1.1}{.11} = 10$$

$$A_x = 1 - d\ddot{a}_x = 1 - \frac{.1}{1.1}(10) = \frac{1}{11}$$

$$\therefore P_x = \frac{A_x}{\ddot{a}_x} = \frac{1/11}{10} = \frac{1}{110}$$

$\therefore \pi =$ level benefit premium $= 1000 P_x$

$$= \frac{1000}{110}$$

$$= \underline{\underline{9.091}}$$

Question No. 2:

The expense-loaded annual premium for a 30-year endowment policy of \$10,000 issued to (30) is to be computed based on the following assumptions:

- sales commission is 40% of the gross premium in the first year;
- renewal commissions are 5% of the gross premium in years 2 through 10 only, and 0% thereafter;
- per policy expenses are \$12.50 per \$1,000 of benefit in the first year and \$2.50 per \$1,000 of benefit thereafter;
- $i = 4\%$; and
- the following table of annuity values:

n	$\ddot{a}_{30:\overline{n}\rceil}$
10	8.409
20	14.010
30	17.626

Calculate the $\underbrace{\text{expense-loaded annual premium}}_{\text{let this be } G}$.

$$G\ddot{a}_{30:\overline{30}\rceil} = \underbrace{10000\,A_{30:\overline{30}\rceil}}_{\text{APV(Benefits)}} + \underbrace{.35G + .05G\ddot{a}_{30:\overline{10}\rceil} + 100 + 25\ddot{a}_{30:\overline{30}\rceil}}_{\text{APV(Expenses)}}$$

$$A_{30:\overline{30}\rceil} = 1 - d\,\ddot{a}_{30:\overline{30}\rceil}$$

$$= 1 - \frac{.04}{1.04}(17.626) = .3221$$

$$\therefore\ G\ddot{a}_{30:\overline{30}\rceil} - .35G - .05G\ddot{a}_{30:\overline{10}\rceil} = 10000\,A_{30:\overline{30}\rceil} + \underbrace{100 + 25\ddot{a}_{30:\overline{30}\rceil}}_{376.65}$$

$$\therefore\ G = \frac{10000(.3221) + 100 + 25(17.626)}{17.626 - .35 - .05(8.409)} = \underline{223.16}$$

16.85555

Question No. 3:

For a fully continuous whole life insurance of 1 on (x), you are given that:

- the force of mortality is constant;
- the force of interest is also constant;
- ${}^2\bar{A}_x = 0.20$; and
- $\bar{P}(\bar{A}_x) = 0.04$;

Calculate $\mathrm{Var}(L)$, where L is the loss-at-issue random variable that is based on the benefit premium.

$$\bar{P}(\bar{A}_x) = \mu = .04$$

$${}^2\bar{A}_x = \frac{\mu}{\mu + 2\delta} = \frac{.04}{.04 + 2\delta} = 0.20 \Rightarrow \delta = 0.08$$

$$\therefore \bar{A}_x = \frac{\mu}{\mu + \delta} = \frac{.04}{.04 + .08} = \frac{1}{3}$$

$$\mathrm{Var}(L) = \frac{{}^2\bar{A}_x - \bar{A}_x^{\,2}}{(1 - \bar{A}_x)^2} = \frac{\frac{1}{5} - \frac{1}{9}}{\frac{4}{9}} = \frac{4/45}{4/9} = \frac{1}{5} = \underline{\underline{0.20}}$$

Question No. 4:

Ben is currently 35 years old and is contemplating whether to purchase a whole life insurance of $100,000 now or wait for another year. Assume the following:

- The death benefit will be paid at the end of the year of death and level premiums are payable once at the beginning of each year.
- The insurer calculates premiums based on $i = 4\%$, $q_{35} = 0.000689$, and $A_{35} = 0.19219$.
- Ignore expenses.

How much more will Ben have to pay in each year if he waits to purchase the insurance when he reaches age 36?

$$A_{35} = v q_{35} + v p_{35} A_{36} \Rightarrow A_{36} = \frac{A_{35}(1+i) - q_{35}}{p_{35}}$$

$$= \frac{.19219(1.04) - .000689}{1 - .000689} = 0.19933$$

$$\ddot{a}_{35} = \frac{1 - A_{35}}{d} = \frac{1 - .19219}{.04/1.04} = 21.003$$

$$\ddot{a}_{36} = \frac{1 - A_{36}}{d} = \frac{1 - .19933}{.04/1.04} = 20.818$$

$$P_{35} = \frac{A_{35}}{\ddot{a}_{35}} = \frac{.19219}{21.003} = .00915$$

$$P_{36} = \frac{A_{36}}{\ddot{a}_{36}} = \frac{.19933}{20.818} = .00957$$

The difference in premium then is

$$100000 * (P_{36} - P_{35}) = 100000(.00957 - .00915)$$

$$= \underline{\underline{42.43}}$$

Question No. 5:

Two actuaries, Kurt and Keisha, use the same mortality table to price a fully discrete two-year endowment of \$10 on (x). You are given:

- Kurt calculates level annual benefit premiums of π.
- Keisha calculates non-level benefit premiums of 5.0 in the first year and 3.7 in the second year.
- $v = 0.90 \implies d = 0.10$

Calculate the value of π.

$$\pi = 10\,P_{x:\overline{2|}} = 10\,\frac{A_{x:\overline{2|}}}{\ddot{a}_{x:\overline{2|}}} = 10\left(\frac{1 - d\,\ddot{a}_{x:\overline{2|}}}{\ddot{a}_{x:\overline{2|}}}\right) = 10\left(\frac{1}{\ddot{a}_{x:\overline{2|}}} - d\right)$$

Keisha's calculations:

$$APV(\text{benefits}) = APV(\text{premiums})$$

$$10\,\underbrace{A_{x:\overline{2|}}}_{\left[1 - \underbrace{d}_{0.10}\,\underbrace{\ddot{a}_{x:\overline{2|}}}_{(1+\underbrace{v}_{0.9}\,p_x)}\right]} = 5 + 3.7\,\underbrace{v}_{0.90}\,p_x$$

$$\implies 10(.9 - .09\,p_x) = 5 + 3.7(.9)\,p_x$$

$$9 - 5 = (3.33 + .9)\,p_x$$

$$p_x = \frac{4}{4.23} = .946$$

$$\therefore\ \ddot{a}_{x:\overline{2|}} = 1 + v\,p_x = 1 + .9(.946) = 1.8510$$

$$\pi = 10\left(\frac{1}{1.8510} - .10\right) = 4.4$$

Question No. 6:

For a whole life insurance of $5,000 issued to (40), you are given:

- Death benefit is payable at the end of the year of death.
- Premiums are level payable annually at the beginning of each year.
- Mortality follows the *Illustrative Life table* with $i = 6\%$.
- Expenses are the following:
  - taxes: 2.5% of the gross annual premium
  - commissions: 4% of the gross annual premium
  - fixed expenses (per $1,000 of death benefit): $5 in the first year and $2.50 in renewal years

Calculate the amount of the $\underbrace{\text{gross annual premium}}_{\text{let this be } G}$.

$$\therefore \quad APV(\text{Premiums}) = APV(\text{Benefits}) + APV(\text{Expenses})$$

$$G\ddot{a}_{40} = 5000 A_{40} + .025 G \ddot{a}_{40} + .04 G \ddot{a}_{40} + 2.5(5) + 2.5(5)\ddot{a}_{40}$$

$$G(1-.065)\ddot{a}_{40} = 5000 A_{40} + 12.5 + 12.5 \ddot{a}_{40}$$

$$G = \frac{5(161.32) + 12.5 + 12.5(14.8166)}{.935(14.8166)}$$

$$= \underline{\underline{72.4947}}$$

Question No. 7:

You are given the following:

- $i = 3\%$
- $A_{30} = 0.27921$
- $\ddot{a}_{40} = 21.68$
- ${}_{10}E_{30} = 0.73756$

$$= \frac{\bar{A}_{30}}{\ddot{a}^{(12)}_{30:\overline{10}|}}$$

Estimate the value of ${}_{10}P^{(12)}(\bar{A}_{30})$ and interpret this actuarial symbol.

$$\bar{A}_{30} \approx \frac{i}{\delta} A_{30} = \frac{.03}{\log(1.03)}(.27921) = 0.28338$$

$$\ddot{a}^{(12)}_{30:\overline{10}|} = \ddot{a}^{(12)}_{30} - {}_{10}E_{30}\, \ddot{a}^{(12)}_{40}$$

$$\ddot{a}^{(12)}_{40} \approx \ddot{a}_{40} - \frac{11}{24} = 21.68 - \frac{11}{24} = 21.22167$$

$$\ddot{a}^{(12)}_{30} \approx \ddot{a}_{30} - \frac{11}{24} = \frac{1 - A_{30}}{d} - \frac{11}{24} = \frac{1 - .27921}{.03/1.03} - \frac{11}{24}$$

$$= 24.28879$$

$$\ddot{a}^{(12)}_{30:\overline{10}|} \approx 24.28879 - 0.73756(21.22167)$$

$$= 8.63654$$

$$\therefore {}_{10}P^{(12)}(\bar{A}_{30}) = \frac{0.28338}{8.63654} = \underline{.0328}$$

this gives the annual benefit premium for a whole life policy that pays benefit at moment of death and where the premium is payable monthly for 10 years.

Question No. 8:

For a fully continuous whole life policy of $1 issued to (45), you are given:

- Force of mortality is constant with $\mu = 0.01$.
- Force of interest is constant with $\delta = 0.05$.
- Premium is determined according to the actuarial equivalence principle.

Calculate the probability that the insurer will make a profit, i.e. the loss-at-issue is negative.

Premium is $\bar{P}(\bar{A}_{45}) = \mu = .01$, independent of interest

Since force of mortality is constant, $T_{45} \sim$ Exponential with $\mu = .01$

$f_{T_{45}}(t) = .01 e^{-.01t}, t > 0$

$$\therefore P(L<0) = P\left(v^T - \pi \bar{a}_{\overline{T|}} < 0\right)$$

$$= P\left(v^T\left(1 + \frac{.01}{.05}\right) - \frac{.01}{.05} < 0\right)$$

$$= P\left(\underset{e^{-\delta T}}{v^T} < \frac{.01/.05}{1.06/.05} = \frac{1}{6}\right)$$

$$= P\left(e^{-.05T} < 6^{-1}\right) = P(.05T > \log 6)$$

$$= P\left(T > \frac{\log 6}{.05}\right)$$

$$= e^{-.01(\log 6/.05)}$$

$$= \underline{\underline{.6988}}$$

Question No. 9:

You are given the following information:

- $i = 5\%$
- $P_{30} = 0.0525$
- $A^{\;1}_{30:\overline{20|}} = 0.05$
- ${}_{20}E_{30} = 0.50$

this is the 20th year benefit reserve for a fully discrete whole life issued to (30)

Calculate ${}_{20}V_{30}$ and interpret this actuarial symbol.

Given the information, best to use retrospective formula

$${}_{20}V_{30} = AAV(\text{Past Premiums}) - AAV(\text{Past Benefits})$$

$$= \frac{\overset{.0525}{P_{30}}\,\ddot{a}_{30:\overline{20|}}}{\underset{.5}{{}_{20}E_{30}}} - \frac{\overset{.05}{A^{\;1}_{30:\overline{20|}}}}{\underset{.5}{{}_{20}E_{30}}}$$

$$\ddot{a}_{30:\overline{20|}} = \frac{1 - A_{30:\overline{20|}}}{d} = \frac{1-(A^{\;1}_{30:\overline{20|}} + {}_{20}E_{30})}{i/1+i}$$

$$= \frac{1-(.05+.5)}{.05/1.05} = 9.45$$

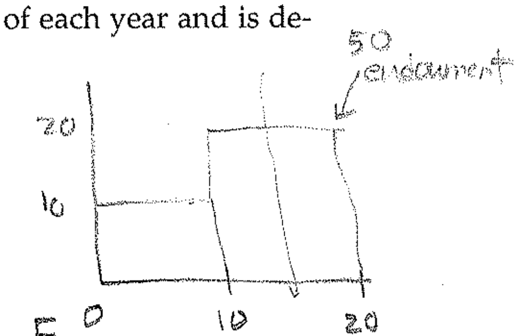
$$\therefore\ {}_{20}V_{30} = \frac{.0525(9.45) - .05}{.5} = \underline{\underline{.89225}}$$

Question No. 10:

For a special 20-year endowment policy issued to age 45, you are given:

- Death benefit is payable at the end of the year of death with benefit amount equal to:
  - \$10 if death is within the first 10 years,
  - \$20 if death is within the next 10 years, and
  - \$50 if alive at the end of 20 years.
- Mortality follows the *Illustrative Life table* with $i = 6\%$.
- The level annual benefit premium is payable at the beginning of each year and is determined according to the actuarial equivalence principle.

Using a prospective formula, calculate the 15th year benefit reserve.



Let π = net annual premium

$$= \frac{10\, A^{1}_{45:\overline{20|}} + 10\, {}_{10}E_{45}\, A^{1}_{55:\overline{10|}} + 50\, {}_{20}E_{45}}{\ddot{a}_{45:\overline{20|}}}$$

$$A^{1}_{45:\overline{20|}} = A_{45} - {}_{20}E_{45}\, A_{65} = .20120 - .25634(.43980)$$
$$= .0885$$

$$A^{1}_{55:\overline{10|}} = A_{55} - {}_{10}E_{55}\, A_{65} = .30514 - .48686(.43980)$$
$$= .0910$$

$$\ddot{a}_{45:\overline{20|}} = \ddot{a}_{45} - {}_{20}E_{45}\, \ddot{a}_{65} = 14.1121 - .25634(9.8969)$$
$$= 11.5751$$

$$\Rightarrow \pi = \frac{10(.0885) + 10(.52652)(.0910) + 50(.25634)}{11.5751} = \underline{1.2251}$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK

Continued (Question 10)

$$.68756$$

$$_{15}V = APV(FB) - APV(FP)$$

$$= \left[20\, A^{1}_{60:\overline{5}|} + 50\, {}_5E_{60} \right] - \pi\, \ddot{a}_{60:\overline{5}|}$$

$$A_{60} - {}_5E_{60} A_{65} = .36913 - .68756(.43980) = .0667$$

$$\ddot{a}_{60:\overline{5}|} = 11.1454 - .68756(9.8969) = 4.3407$$

$$_{15}V = \left[20(.0667) + 50(.68756)\right] - 1.2251(4.3407)$$

$$= \underline{\underline{30.395}}$$

Question No. 11:

You are given the following:

- ${}_{15}P_{45} = 0.038$;
- $P_{45:\overline{15}|} = 0.056$; and
- $A_{60} = 0.0625$.

this is the annual benefit premium for a 15-year term insurance of $1 issued to (45).

Calculate $P^{\ 1}_{45:\overline{15}|}$ and interpret this value.

$$P^{\ 1}_{45:\overline{15}|} = \frac{A^{\ 1}_{45:\overline{15}|}}{\ddot{a}_{45:\overline{15}|}} = \underbrace{\frac{A_{45:\overline{15}|}}{\ddot{a}_{45:\overline{15}|}}}_{P_{45:\overline{15}|} = .056} - \frac{{}_{15}E_{45}}{\ddot{a}_{45:\overline{15}|}}$$

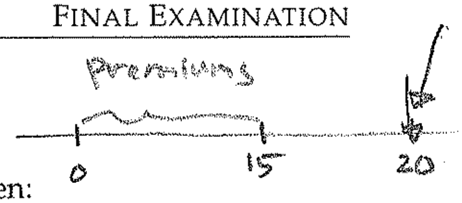
$${}_{15}P_{45} = \frac{A_{45}}{\ddot{a}_{45:\overline{15}|}} = \frac{A^{\ 1}_{45:\overline{15}|} + {}_{15}E_{45} A_{60}}{\ddot{a}_{45:\overline{15}|}}$$

$$= P^{\ 1}_{45:\overline{15}|} + \underbrace{A_{60}}_{.0625} \left(\frac{{}_{15}E_{45}}{\ddot{a}_{45:\overline{15}|}} \right) = .038$$

$$\therefore \frac{{}_{15}E_{45}}{\ddot{a}_{45:\overline{15}|}} = \frac{.038 - P^{\ 1}_{45:\overline{15}|}}{.0625}$$

$$P^{\ 1}_{45:\overline{15}|} = .056 - \left(\frac{.038 - P^{\ 1}_{45:\overline{15}|}}{.0625} \right)$$

Solving for $P^{\ 1}_{45:\overline{15}|}$, we get $P^{\ 1}_{45:\overline{15}|} = \underline{.0368}$



Question No. 12:

For a whole life insurance policy of $1 issued to (40), you are given:

- Benefit is payable at the end of the year of death.
- Level premium is payable at the beginning of each year for 15 years.

Using actuarial symbols you have learned from this course, give both the prospective and the retrospective benefit reserve formulas at the end of 20 years.

Prospectively: $${}^{15}_{20}V_{40} = APV(FB)$$ since no more premiums left to pay

$$= A_{60}$$

Retrospectively: $${}^{15}_{20}V_{40} = AAV(PP) - AAV(PB)$$

$$= \frac{{}_{15}P_{40}\, \ddot{a}_{40:\overline{15|}}}{{}_{20}E_{40}} - \frac{A^{1}_{40:\overline{20|}}}{{}_{20}E_{40}}$$

Question No. 13:

You are given the following table of annuity values:

x	$\ddot{a}_x$
30	18.5
40	13.8
50	6.6

Calculate ${}_{10}V_{30}$.

$${}_{10}V_{30} = 1 - \frac{\ddot{a}_{40}}{\ddot{a}_{30}} = 1 - \frac{13.8}{18.5}$$

$$= \underline{\underline{.2541}}$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK