MATH 3630
Actuarial Mathematics I
Final Examination
Wednesday, 14 December 2011
Time Allowed: 2 hours (3:30 - 5:30 pm)
Room: MSB 411
Total Marks: 120 points

Please write your name and student number at the spaces provided:

Name: EMIL **Student ID:** SUGGESTED SOLUTIONS

- There are twelve (12) written-answer questions here and you are to answer all twelve. Each question is worth 10 points. Your final mark will be divided by 120 to convert to a unit of 100%.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught writing after time has expired will be given a mark of zero.
- Good luck.
- Have a Happy and Healthy Christmas and New Year!

Question	Worth	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120	
%	$\div$ 120	

Question No. 1:

Assume the following:

- Mortality follows Gompertz mortality law, $\mu_x = Bc^x$, where

$$B = 0.0001 \quad \text{and} \quad c = 1.09.$$

- Interest rate is $i = 4\%$.

Calculate $100P_{60:\overline{2}|}$ and interpret this value.

Recall for Gompertz $p_x = e^{-\frac{B}{\log c} c^x (c-1)}$

$$\ddot{a}_{60:\overline{2}|} = 1 + v\, p_{60} = 1 + (1.04)^{-1} \underbrace{e^{-\frac{.0001}{\log(1.09)} (1.09)^{60} (1.09-1)}}_{.9817841}$$

$$= 1.944023$$

$$A_{60:\overline{2}|} = 1 - d\, \ddot{a}_{60:\overline{2}|} = 1 - (1 - (1.04)^{-1})(1.944023)$$

$$= 0.9252299$$

$$100\, P_{60:\overline{2}|} = 100 \frac{A_{60:\overline{2}|}}{\ddot{a}_{60:\overline{2}|}} = 100 \frac{(.9252299)}{1.944023} = \underline{\underline{47.59356}}$$

This gives the annual benefit premium of a two-year endowment policy of 100 to (60).

Question No. 2:

$${}_tp_x = e^{-At}\, e^{-\frac{B}{\log c} c^x (c^t - 1)}$$

You are given:

- Mortality follows Makeham's law with $A = 0.005$, $B = 0.00001$ and $c = 1.12$.
- $i = 0.05$
- $\ddot{a}_{65} = 10.3351$

Calculate ${}_{10|}\ddot{a}_{55}^{(12)}$ based on Woolhouse's approximation with three terms.

$${}_{10|}\ddot{a}_{55}^{(12)} = {}_{10}E_{55}\, \ddot{a}_{65}^{(12)}$$

where

$$\ddot{a}_{65}^{(12)}\ W3 = \underbrace{\ddot{a}_{65}}_{10.3351} - \frac{12-1}{2(12)} - \frac{12^2-1}{12(12^2)}\left(\mu_{65} + \delta\right)$$

$$\mu_{65} = .005 + .00001\,(1.12)^{65} \qquad \delta = \log(1.05)$$

$$= 9.871006$$

$${}_{10}E_{55} = v^{10}\, {}_{10}p_{55} = \underbrace{(1.05)^{-10}\, e^{-.005(10)}\, e^{-\frac{.00001}{\log(1.12)}(1.12)^{55}\left(1.12^{10} - 1\right)}}$$

$$= 0.5312394$$

$${}_{10|}\ddot{a}_{55}^{(12)}\ W3 = {}_{10}E_{55}\, \ddot{a}_{65}^{(12)}\ W3$$

$$= (.5312394)(9.871006)$$

$$= \underline{\underline{5.243867}}$$

Question No. 3:

Mr. Jack Luck, now age 40, wins 1,000,000 in a lottery.

Rather than receiving his winnings as a lump sum, he is offered the actuarially equivalent option of receiving an annual payment of b (at the beginning of each year) guaranteed for 10 years and continuing thereafter so long as he is alive.

You are given:

- $A_{40} = 0.291$
- $A_{50} = 0.414$
- $A^{\ 1}_{40:\overline{10|}} = 0.017$
- $i = 4\%$

Calculate b.

$$1{,}000{,}000 = b\left(\ddot{a}_{\overline{10|}} + {}_{10}E_{40}\,\ddot{a}_{50}\right)$$

$$b = \frac{1{,}000{,}000}{\ddot{a}_{\overline{10|}} + {}_{10}E_{40}\,\ddot{a}_{50}}$$

where $\ddot{a}_{\overline{10|}} = \dfrac{1-v^{10}}{d} = \dfrac{1-(1.04)^{-10}}{.04/1.04} = 8.435332$

$$A_{40} = A^{\ 1}_{40:\overline{10|}} + {}_{10}E_{40}\,A_{50}$$

$$.291 = .017 + {}_{10}E_{40}\,(.414) \Rightarrow {}_{10}E_{40} = \frac{.291-.017}{.414} = .6618357$$

$$\ddot{a}_{50} = \frac{1-A_{50}}{d} = \frac{1-.414}{.04/1.04} = 15.236$$

$$b = \frac{1000000}{8.435332 + (.6618357)(15.236)} = \underline{\underline{53{,}998.42}}$$

Question No. 4:

For a fully discrete whole life insurance of \$100 issued to (50), you are given:

- Mortality follows the `Illustrative Life Table` with $i = 0.06$.
- Premium is calculated according to the equivalence principle.
- L_0 denotes the loss-at-issue random variable for this policy.

Calculate $\text{Var}[L_0]$.

$$\text{Var}[L_0] = 100^2\left(1 + \frac{P_{50}}{d}\right)^2\left({}^2A_{50} - (A_{50})^2\right)$$

where $P_{50} = \dfrac{A_{50}}{\ddot{a}_{50}} = \dfrac{.24905}{13.2668} = 0.01877242$

$d = 1 - (1.06)^{-1}$

${}^2A_{50} = .09476$

Plus the values to get,

$$\text{Var}[L_0] = \underbrace{100^2\left(1 + \frac{.01877242}{1-(1.06)^{-1}}\right)^2\left[.09476 - (.24905)^2\right]}_{}$$

$$\underline{\underline{580.4677}}$$

Question No. 5:

Vis-ta-Vie Life Insurance Company sells a portfolio of fully discrete whole life insurance of \$100 to individuals age 40, something it has not done before. The actuaries of the company have determined that:

- For each policy, the annual benefit premium will be \$0.80.
- Mortality follows the `Standard Ultimate Survival Model` with $i = 5\%$.
- All the policyholders have independent future lifetimes.

Using Normal approximation, calculate the smallest number of policies the company must sell to ensure that the probability of a positive loss at issue, on the aggregate, does not exceed 0.05.

Let m be the number of policies to sell so that the aggregate loss at issue is $L_0 = L_{0,1} + L_{0,2} + \dots + L_{0,m}$

where

$$L_{0,i} = 100 v^{K+1} - .80\, \ddot{a}_{\overline{K+1}|} \qquad \frac{1-v^{K+1}}{d}$$

$$= \left(100 + \frac{.80}{d}\right) v^{K+1} - \frac{.80}{d}, \quad i = 1, 2, \dots, m$$

$$E[L_{0,i}] = \left(100 + \frac{.80}{d}\right) A_{40} - \frac{.80}{d} = -2.660192$$

$A_{40} = .12106$, $d = .05/1.05$

$$Var[L_{0,i}] = \left(100 + \frac{.80}{d}\right)^2 \left({}^2A_{40} - A_{40}^2\right)$$

${}^2A_{40} = .02347$, $A_{40}^2 = (.12106)^2$

$$= 120.2492$$

THIS PAGE FOR EXTRA SPACE TO SOLVE QUESTION 5

$$\Pr[L_0 > 0] \le .05 \Rightarrow \Pr\left[Z > \frac{2.660192\, m}{\sqrt{120.2492\, m}}\right] \le .05$$

Thus,

$$\frac{2.660192\, \cancel{m}^{\sqrt{m}}}{\sqrt{120.2492\, \cancel{m}}} \ge 1.645$$

$$\sqrt{m} \ge \frac{1.645\sqrt{120.2492}}{2.660192} = 6.781004$$

$$m \ge 45.98202$$

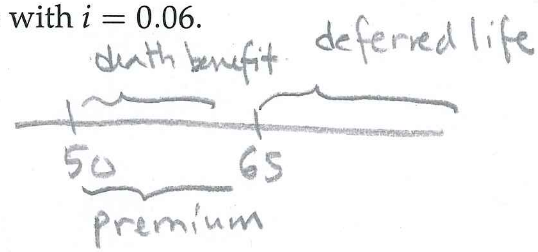
Company must sell at least 46 policies!

Question No. 6:

Get-a-Life Insurance Company issues a 15-year deferred life annuity contract to a policyholder now age 50. You are given:

- Level annual benefit premiums of P are paid during the deferred period.
- The annuity benefit of \$500 is to be paid at the beginning of each year the policyholder is alive, with the first payment to be made when he reaches the age of 65.
- An additional death benefit of \$10,000 is to be paid if at the end of the year of death, if death occurs during the deferred period.
- Mortality follows the `Illustrative Life Table` with $i = 0.06$.

Calculate P.



APV(premiums) = APV(benefits)

$$P\ddot{a}_{50:\overline{15}|} = 10{,}000\, A^{\,1}_{50:\overline{15}|} + 500\, {}_{15}E_{50}\, \ddot{a}_{65}$$

where

$$\ddot{a}_{50:\overline{15}|} = \underbrace{\ddot{a}_{50}}_{13.2668} - {}_{15}E_{50}\, \overset{9.8969}{\ddot{a}_{65}} = 9.790947$$

$${}_{15}E_{50} = \underbrace{{}_{5}E_{50}}_{.72137}\ \overset{.48686}{{}_{10}E_{55}} \qquad {}_{15}E_{50} = .3512062$$

$$A^{\,1}_{50:\overline{15}|} = \underbrace{A_{50}}_{.24905} - {}_{15}E_{50}\, \overset{.43980}{A_{65}} = .09458951$$

$$P = \frac{10{,}000\,(.09458951) + 500\,(.3512062)(9.8969)}{9.790947}$$

$$= \frac{2683.821}{9.790947} = \underline{\underline{274.1125}}$$

Question No. 7:

For a a fully continuous whole life insurance of \$1 issued to (x), you are given:

- $\mu_{x+t} = 0.01$, for all $t \geq 0$
- $\delta = 0.04$
- Annual benefit premium is set at $1.10P$ where P is the annual benefit premium calculated based on the equivalence principle.
- L_0 is the loss-at-issue random variable.

Calculate $\mathrm{Var}[L_0]$.

$$L_0 = v^T - 1.10P\bar{a}_{\overline{T|}} = \left(1+\frac{1.10P}{\delta}\right)v^T - \frac{1.10P}{\delta}$$

$$\mathrm{Var}[L_0] = \left(1+\frac{1.10P}{\delta}\right)^2 \underbrace{\mathrm{Var}[v^T]}_{{}^2\bar{A}_x - \bar{A}_x^2}$$

$\delta = .04$

$1.10P = 1.10(.01) = .011$

$$\bar{A}_x = \frac{\mu}{\mu+\delta} = \frac{.01}{.05} = .20$$

$$^2\bar{A}_x = \frac{\mu}{\mu+2\delta} = \frac{.01}{.09} = \frac{1}{9} = .111111$$

$$\mathrm{Var}[L_0] = \left(1+\frac{.011}{.04}\right)^2 (.111111 - (.20)^2)$$

$$= \underline{\underline{0.1156}}$$

Question No. 8:

Mr. Tu Kuhl is currently age 50 who purchases a deferred whole life annuity-due policy which will pay him the following benefits:

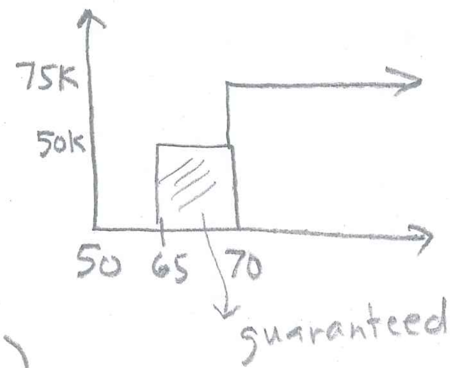
- guaranteed annual payments of \$50,000 for 5 years, with the first payment to start when he reaches age 65; and
- annual payments of \$75,000 thereafter, if alive.

You are given:

- Mortality follows the `Illustrative Life Table`.
- $i = 6\%$

Calculate the actuarial present value of Mr. Kuhl's life annuity benefits.

$$\text{APV(annuity)} = 25{,}000 * \left(2\,{}_{15}E_{50}\,\ddot{a}_{\overline{5}|} + 3\,{}_{20}E_{50}\,\ddot{a}_{70} \right)$$



$$= 25{,}000 * \left(2(.3512062)\left(\frac{1-(1.06)^{-5}}{1-(1.06)^{-1}}\right) + 3(.23047)(8.5693) \right)$$

$$= \$226{,}531.10$$

Question No. 9:

For a fully discrete whole life insurance of \$1 issued to (40), you are given:

- P is the annual benefit premium determined according to the equivalence principle.
- P^* is the smallest possible annual benefit premium to ensure that the probability of a positive loss-at-issue is less than 0.50.

You are given:

- Mortality follows the `Illustrative Life Table`.
- $i = 6\%$

Calculate $\frac{P}{P^*}$.

$$P = \frac{A_{40}}{\ddot{a}_{40}} = \frac{.16132}{14.8166} = .01088779$$

$$\Pr[L_0(P^*) > 0] \leq 0.50 \qquad K = K_{40} = \text{curtate future lifetime of } (40)$$

$$L_0(P^*) = v^{K+1} - \frac{P^*}{d}(1 - v^{K+1})$$

$$= \left(1 + \frac{P^*}{d}\right) v^{K+1} - \frac{P^*}{d} > 0$$

$$\Leftrightarrow K < \underbrace{-1 - \frac{1}{\delta} \log\left(\frac{P^*/d}{1 + P^*/d}\right)}_{m}$$

$$\Leftrightarrow \Pr[K < m] \leq 0.50 \Leftrightarrow {}_mP_{40} \geq 0.50 \Leftrightarrow \frac{l_{40+m}}{l_{40}} \geq 0.50$$

$$\Leftrightarrow l_{40+m} \geq 0.50\, l_{40} \overset{9313166}{} = 4,656,583$$

$$l_{77} = 4,828,182$$
$$l_{78} = 4,530,360 \quad \Rightarrow \quad m \text{ must be at least } 38$$

$$-1 - \frac{1}{\delta} \log\left(\frac{P^*/d}{1 + P^*/d}\right) \leq 38 \Leftrightarrow P^* \geq \frac{e^{-39\delta}}{1 - e^{-39\delta}} \cdot d = .00650355\overline{9}$$

$$\frac{P}{P^*} = \frac{.01088779}{.006503559} = \underline{\underline{1.674128}}$$

Question No. 10:

You are given:

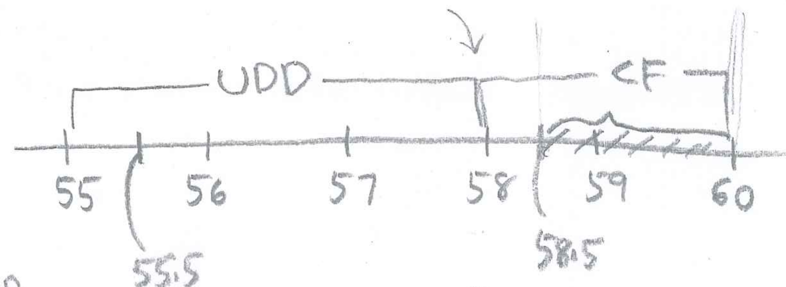
- the following extract from a mortality table:

x	55	56	57	58	59	60
ℓ_x	10,000	9,950	9,900	9,850	9,800	9,750

- Mortality follows UDD assumption over each year of age between ages 55 and 58.
- Mortality follows constant force assumption over each year of age between ages 58 and 60.

Calculate the probability that an individual now age 55.5 will die between ages 58.5 and 60.

CF = constant force



$${}_{3|1.5}q_{55.5} = {}_{2.5}p^{UDD}_{55.5} * \left[{}_{.5}p^{CF}_{58} - {}_{2}p^{CF}_{58} \right]$$

$$= \frac{\cancel{\ell_{58}}}{\ell^{UDD}_{55.5}} * \left[\frac{\ell^{CF}_{58.5} - \ell_{60}}{\cancel{\ell_{58}}} \right]$$

$$= \frac{\ell_{58}^{1/2}\,\ell_{59}^{1/2} - \ell_{60}}{\frac{1}{2}(\ell_{55} + \ell_{56})} = \frac{(9850)^{1/2}(9800)^{1/2} - 9750}{\frac{1}{2}(10,000 + 9950)}$$

$$= \frac{74.96819}{9975} = .007515608$$

Question No. 11:

You are given:

- $P_{x:\overline{n}|} = 0.0504$
- $P^{1}_{x:\overline{n}|} = 0.0056$
- ${}_nP_x = 0.0292$

Calculate A_{x+n}.

$$A_x = A^{1}_{x:\overline{n}|} + {}_nE_x A_{x+n}$$

divide both sides by $\ddot{a}_{x:\overline{n}|}$

$$\underbrace{\frac{A_x}{\ddot{a}_{x:\overline{n}|}}}_{{}_nP_x} = \underbrace{\frac{A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}}_{P^{1}_{x:\overline{n}|}} + \underbrace{\frac{(A_{x:\overline{n}|} - A^{1}_{x:\overline{n}|})}{\ddot{a}_{x:\overline{n}|}}}_{(P_{x:\overline{n}|} - P^{1}_{x:\overline{n}|})} A_{x+n}$$

Solving for A_{x+n},

$$A_{x+n} = \frac{{}_nP_x - P^{1}_{x:\overline{n}|}}{P_{x:\overline{n}|} - P^{1}_{x:\overline{n}|}} = \frac{.0292 - .0056}{.0504 - .0056}$$

$$= \frac{.0236}{.0448}$$

$$= \underline{\underline{0.5267857}}$$

Question No. 12:

Suppose $i = 5\%$ and you are given the following extract from a select-and-ultimate mortality table:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	x+3
40	75000	74511	73874	73123	43
41	74171	73668	73017	72245	44
42	73318	72801	72132	71341	45
43	72444	71910	71223	70409	46
44	71545	70994	70286	69448	47
45	70616	70046	69315	68451	48

Calculate $P_{[42]:\overline{3}|}$.

$$\ddot{a}_{[42]:\overline{3}|} = 1 + v\, p_{[42]} + v^2\, p_{[42]}\, p_{[42]+1}$$

$$= 1 + v \frac{\ell_{[42]+1}}{\ell_{[42]}} + v^2 \frac{\ell_{[42]+2}}{\ell_{[42]}}$$

$$= 1 + (1.05)^{-1}\left(\frac{72801}{73318}\right) + (1.05)^{-2}\left(\frac{72132}{73318}\right)$$

$$= 2.838023$$

$$A_{[42]:\overline{3}|} = 1 - (1-(1.05)^{-1})(2.838023)$$

$$= 0.8648561$$

$$P_{[42]:\overline{3}|} = \frac{A_{[42]:\overline{3}|}}{\ddot{a}_{[42]:\overline{3}|}} = \frac{0.8648561}{2.838023} = \underline{\underline{0.304739}}$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK