MATH 3630
Actuarial Mathematics I
Sample Test 1
Time Allowed: 1 hour
Total Marks: 100 points

Please write your name and student number at the spaces provided:

Name: SUGGESTED SOLUTIONS **Student ID:** EMIL

- **There are ten (10) written-answer questions here and you are to answer all ten. Each question is worth 10 points.**
- **Please provide details of your workings in the appropriate spaces provided; partial points will be granted.**
- **Please write legibly.**
- **Anyone caught writing after time has expired will be given a mark of zero.**

Question No. 1:

Let X be the age-at-death random variable.

Assume X follows deMoivre's law with $\omega = 100$.

Calculate ${}_{10}m_{20}$.

$X \sim$ deMoivre's implies Uniform on $(0, 100)$. Thus

$$S_X(x) = 1 - \frac{x}{100}, \quad 0 \le x \le 100$$

$$\Rightarrow \mu_x = \frac{-1}{S_X(x)} \frac{d}{dx} S_X(x) = \frac{1/\cancel{100}}{\frac{100-x}{\cancel{100}}} = \frac{1}{100-x}.$$

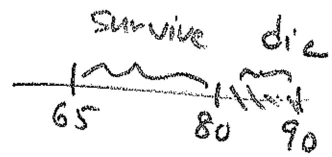
$$\therefore \quad {}_{10}m_{20} = \frac{\int_{20}^{30} S_X(y)\mu_y \, dy}{\int_{20}^{30} S_X(y)\, dy} = \frac{\int_{20}^{30} \left(1 - \frac{y}{100}\right) \frac{1}{100-y}\, dy}{\int_{20}^{30} \left(1 - \frac{y}{100}\right) dy}$$

$$= \frac{\frac{1}{100}(30-20)}{10 - \frac{1}{200}(30^2 - 20^2)} = \frac{10/\cancel{100}}{10 - 5/2} = \frac{1}{75}.$$

Question No. 2:

Assume that mortality follows the *Illustrative Life Table*.

Calculate the probability that a life (65) will die between ages 80 and 90.



The required probability is $_{15|10}q_{65}$.

$$_{15|10}q_{65} = \frac{l_{80} - l_{90}}{l_{65}}$$

$$= \frac{3914365 - 1058491}{7533964}$$

$$= \underline{\underline{0.3791}}$$

Question No. 3:

You are given:

$$_{k|}q_0 = 0.1(k+1), \quad \text{for } k = 0, 1, 2 \text{ and } 3.$$

Suppose UDD holds between integral ages.

Compute the value of $_{2.75}p_0$ and interpret this probability.

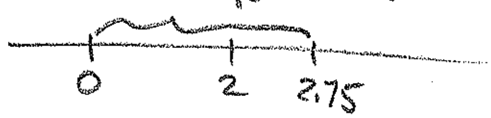
First, compute the probabilities for $K = 0, 1, 2$

$$K=0 \Rightarrow {}_{0|}q_0 = 0.10 = \cancel{{}_0p_0}^{\,1}\, q_0 \Rightarrow q_0 = 0.10 = 1/10$$

$$K=1 \Rightarrow {}_{1|}q_0 = 0.20 = p_0\, q_1 \Rightarrow q_1 = \frac{0.20}{0.90} = 2/9$$

$$K=2 \Rightarrow {}_{2|}q_0 = 0.30 = {}_2p_0\, q_2 \Rightarrow q_2 = \frac{0.30}{\frac{\cancel{9}}{10}\cdot\frac{7}{\cancel{9}}} = 3/7$$

$_{2.75}p_0$ = Probability a newborn survives 2.75 years.



$$_{2.75}p_0 = \underbrace{_2p_0}\ *\ \underbrace{_{.75}p_2}$$

$$= \frac{9}{10}\cdot\frac{7}{9}\left(1 - .75\,q_2\right)$$

$$= \frac{7}{10}\left(1 - \frac{3}{4}\,\frac{3}{7}\right) = \frac{\cancel{7}}{10}\,\frac{19}{\cancel{28}_4} = \frac{19}{40}$$

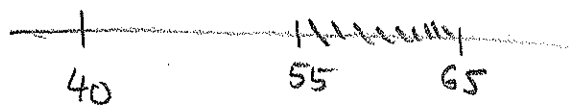
$$= .475$$

Question No. 4:

You are given:

$$\begin{aligned} {}_5p_{40} &= \frac{4}{5} \\ {}_{10}p_{45} &= \frac{3}{5} \\ {}_{10}p_{55} &= \frac{2}{5} \end{aligned}$$

Find the probability that a life (40) will die between ages 55 and 65.



The required probability is ${}_{15|10}q_{40}$.

$$\begin{aligned} {}_{15|10}q_{40} &= \underbrace{{}_{15}p_{40}} * {}_{10}q_{55} \\ &= {}_5p_{40}\ {}_{10}p_{45} * (1 - {}_{10}p_{55}) \\ &= \frac{4}{5}\ \frac{3}{5}\left(1 - \frac{2}{5}\right) \\ &= \frac{36}{125} = \underline{\underline{0.288}} \end{aligned}$$

Question No. 5:

Suppose you are given the survival function:

$$S_X(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha}, \text{ for } 0 \leq x \leq \omega.$$

Prove the following: $\mu_x \cdot \mathring{e}_x = \dfrac{\alpha}{\alpha + 1}$.

$$\mu_x = \frac{-1}{S_x(x)} \frac{d}{dx} S_x(x) = \frac{\cancel{-}1}{\left(1-\frac{x}{\omega}\right)^{\cancel{\alpha}}} \alpha \cancel{\left(1-\frac{x}{\omega}\right)^{\alpha-1}} \left(\frac{\cancel{-}1}{\omega}\right) = \frac{\alpha}{\omega - x}$$

$$\mathring{e}_x = \int_0^{\omega - x} {}_tp_x \, dt = \int_0^{\omega - x} \frac{S_x(x+t)}{S_x(x)} dt$$

$$= \int_0^{\omega-x} \frac{\left(\frac{\omega - x - t}{\cancel{\omega}}\right)^{\alpha}}{\left(\frac{\omega - x}{\cancel{\omega}}\right)^{\alpha}} dt = \int_0^{\omega - x} \left(1 - \frac{t}{\omega - x}\right)^{\alpha} dt$$

$$= -(\omega - x) \frac{\left(1 - \frac{t}{\omega - x}\right)^{\alpha+1}}{\alpha + 1} \Bigg|_0^{\omega - x} = \frac{\omega - x}{\alpha + 1}$$

$$\therefore \mu_x \mathring{e}_x = \frac{\alpha}{\cancel{\omega - x}} \cdot \frac{\cancel{\omega - x}}{\alpha + 1} = \frac{\alpha}{\alpha + 1} \doteq$$

Question No. 6:

Suppose that for an 80-year-old fellow, his force of mortality is given by

$$\mu_{80+t} = \frac{1}{10-t}, \quad \text{for } 0 \le t < 10.$$

Calculate the probability that this fellow will die between ages 85 and 90.

$$_tp_{80} = e^{-\int_0^t \mu_{80+s}\,ds} = e^{-\int_0^t \frac{1}{10-s}\,ds} = 1 - t/10, \quad 0 \le t < 10.$$

$$_tp_{80}\,\mu_{80+t} = f_{T_{80}}(t) = \left(1-\frac{t}{10}\right)\left(\frac{1}{10-t}\right) = \frac{1}{10}$$

$T_{80} \sim$ Uniform on $[0, 10)$.

$\therefore$ Required probability is

$$P(5 < T_{80} < 10) = \int_5^{10} \frac{1}{10}\,dt = \frac{5}{10} = \frac{1}{2} = \underline{\underline{0.5}}$$

Question No. 7:

The following is an extract from a *standard* mortality table:

x	q_x
40	.00278
41	.00298
42	.00320

A *substandard* table is obtained from this *standard* table by adding a constant $c = 0.10$ to the force of mortality. This results in mortality rates denoted by q_x^s, with the superscript s denoting *substandard*.

Calculate the probability that a *substandard* life (40) will die between ages 41 and 42.

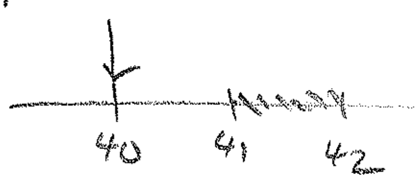
First, derive relationship between standard & substandard. Since mortality is increased by a constant force, work with p's. Thus,

$$p_x^s = e^{-\int_0^1 \mu_{x+t}^s dt} = e^{-\int_0^1 (\mu_{x+t} + .10) dt}$$

$$= e^{-.10} p_x$$

and $q_x^s = 1 - e^{-.10} p_x$.

Required probability is



$$_{1|}q_{40}^s = p_{40}^s * q_{41}^s$$

$$= e^{-.10} p_{40} * (1 - e^{-.10} p_{41})$$

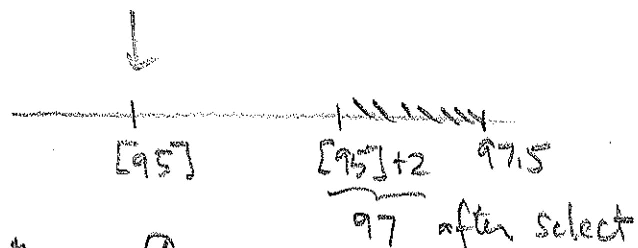
$$= e^{-.10} (.99722)(1 - e^{-.10}(.99702)) = \underline{\underline{.0883}}$$

Question No. 8:

Suppose you are given the following select-and-ultimate mortality table:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}
95	300	60	15
96	175	10	0
97	15	0	0
98	1	0	0

Assuming UDD between integral ages, calculate ${}_{2|0.5}q_{[95]}$.



$$ {}_{2|0.5}q_{[95]} = {}_2p_{[95]} * {}_{0.5}q_{[95]+2} $$

$$ = {}_2p_{[95]} * 0.5 * q_{97} $$

$$ = \frac{l_{97}}{l_{[95]}} * 0.5 * \left(1 - \frac{l_{98}}{l_{97}}\right) $$

$$ = \frac{\cancel{15}}{\underset{20}{\cancel{300}}} * \frac{1}{2} * \left(1 - \frac{\cancel{0}^{\,0}}{15}\right) = \frac{1}{40} = .025 $$

Question No. 9:

Suppose you are given that:

$$\ell_x = 1000\left(27 - \frac{3}{10}x\right)^{1/3}, \text{ for } 0 \le x \le 90.$$

Calculate the average future lifetime of a newborn.

rewrite as $\ell_x = 1000\left(\frac{3}{10}\right)^{1/3}(90-x)^{1/3}$

$$S_X(x) = \frac{\ell_x}{\ell_0} = \frac{1000\left(\frac{3}{10}\right)^{1/3}(90-x)^{1/3}}{\ell_0}$$

$$\therefore\ \mathring{e}_0 = \int_0^{90} S_X(x)\,dx = \frac{1000\left(\frac{3}{10}\right)^{1/3}}{\ell_0} \cdot \int_0^{90} (90-x)^{1/3}\,dx$$

$$= \frac{1000\left(\frac{3}{10}\right)^{1/3}}{\ell_0} \left. \frac{-(90-x)^{4/3}}{4/3} \right|_0^{90}$$

$$= \frac{1000\left(\frac{3}{10}\right)^{1/3}}{\ell_0} \frac{3}{4} 90^{4/3}$$

at $x=0$, $\ell_0 = 1000\left(\frac{3}{10}\right)^{1/3} 90^{1/3} \Rightarrow \mathring{e}_0 = \frac{3}{4}(90)^{\cancel{4/3}} / \cancel{90^{1/3}}$

$$= \frac{270}{4} = \underline{\underline{67.50}}$$

average years to live from birth.

Question No. 10:

You are given the survival function:

$$s_X(x) = \left(1 - \frac{x}{\omega}\right)^{5/2}, \text{ for } 0 \le x \le \omega.$$

If $\mu_{80} = 0.05$, calculate $\mathring{e}_{60:\overline{25}|}$ and interpret this value.

Since $\mu_x = \frac{-1}{S_X(x)} \frac{d}{dx} S_X(x) = \frac{-1}{\left(1-\frac{x}{\omega}\right)^{5/2}} \cdot \frac{5}{2}\left(1-\frac{x}{\omega}\right)^{3/2}\left(-\frac{1}{\omega}\right) = \frac{5}{2}\frac{1}{\omega - x},$

at $x = 80 \Rightarrow \mu_{80} = \frac{5}{2}\frac{1}{\omega-80} = .05 = \frac{5}{100}$

$\Rightarrow \omega - 80 = 50 \Rightarrow \underline{\omega = 130}$

$${}_tp_{60} = \frac{S_X(60+t)}{S_X(60)} = \frac{\left(1-\frac{60+t}{130}\right)^{5/2}}{\left(1-\frac{60}{130}\right)^{5/2}} = \left(\frac{70-t}{70}\right)^{5/2} = \left(1-\frac{t}{70}\right)^{5/2}$$

$$\therefore \mathring{e}_{60:\overline{25}|} = \int_0^{25}\left(1-\frac{t}{70}\right)^{5/2} dt = \frac{-70\left(1-\frac{t}{70}\right)^{7/2}}{7/2}\Bigg|_0^{25}$$

$$= -20\left[\left(1-\frac{5}{14}\right)^{7/2} - 1\right]$$

$$= 20\left[1-\left(\frac{9}{14}\right)^{7/2}\right] = \underline{15.74}$$

This gives number of years (60) is expected to live in the next 25 years.

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK