Life Tables and Selection

Lecture: Weeks 4-5

Chapter summary

- What is a life table?
  - also called a mortality table
  - tabulation of basic mortality functions
  - deriving probabilities/expectations from a life table
- Relationships to survival functions
- Assumptions for fractional (non-integral) ages
- Select and ultimate tables
  - national life tables
  - valuation or pricing tables
- Chapter 3, DHW

What is the life table?

- A tabular presentation of the mortality evolution of a cohort group of lives.
- Begin with ℓ_0 number of lives (e.g. 100,000) - called the radix of the life table.
- (Expected) number of lives who are age x: $\ell_x = \ell_0 \cdot S_0(x) = \ell_0 \cdot {}_xp_0$
- (Expected) number of deaths between ages x and $x+1$: $d_x = \ell_x - \ell_{x+1}$.
- (Expected) number of deaths between ages x and $x+n$: ${}_nd_x = \ell_x - \ell_{x+n}$.
- Conditional on survival to age x, the probability of dying within n years is: ${}_nq_x = {}_nd_x/\ell_x = (\ell_x - \ell_{x+n})/\ell_x$.
- Conditional on survival to age x, the probability of living to reach age $x+n$ is: ${}_np_x = 1 - {}_nq_x = \ell_{x+n}/\ell_x$.

Example of a life table

x	ℓ_x	d_x	q_x	p_x	$\mathring{e}_x$
0	100,000	680	0.006799	0.993201	77.84
1	99,320	48	0.000483	0.999517	77.37
2	99,272	29	0.000297	0.999703	76.41
3	99,243	22	0.000224	0.999776	75.43
⋮	⋮	⋮	⋮	⋮	⋮
50	93,735	413	0.004404	0.995596	30.87
51	93,323	443	0.004750	0.995250	30.01
52	92,879	475	0.005113	0.994887	29.15
53	92,404	507	0.005488	0.994512	28.30
⋮	⋮	⋮	⋮	⋮	⋮
97	5,926	1,370	0.231201	0.768799	3.15
98	4,556	1,133	0.248600	0.751400	2.95
99	3,423	913	0.266786	0.733214	2.76

Source: U.S. Life Table for the total population, 2004, Center for Disease Control and Prevention (CDC)

Radix of the life table

- The radix of the life table does not have to start at age 0, e.g. start with age x_0, so that the table starts with radix ℓ_{x_0}.
- The limiting age of the table is usually denoted by ω, in which case the table ends at $\omega - x_0$.
- All the formulas still work, e.g. conditional on survival to age x, the probability of surviving to reach age $x+n$ is:

$$ {}_n p_x = 1 - {}_n q_x = \frac{\ell_{x+n}}{\ell_x}. $$

- Note that among ℓ_x independent lives who have reached age x, the number of survivors $\mathcal{L}_n$ within n years is a Binomial random variable with parameters ℓ_x and ${}_n p_x$ so that

$$ \mathsf{E}(\mathcal{L}_n) = \ell_x \cdot {}_n p_x. $$

Revised example 3.1

Using Table 3.1, page 43 of DHW, calculate the following:

- the probability that (30) will survive another 5 years
- the probability that (39) will survive to reach age 40
- the probability that (30) will die within 10 years
- the probability that (30) dies between ages 36 and 38

Illustrative example 1

Complete the following life table:

x	ℓ_x	d_x	p_x	q_x
40	24,983	·	·	·
41	24,541	·	·	·
42	24,175	·	·	·
43	23,880	·	·	·
44	23,656	·	·	·
45	23,495	—	—	—

Additional useful formulas

From a life table, the following formulas can also easily be verified (or use your intuition):

- $\ell_x = \sum_{k=0}^{\infty} d_{x+k}$: the number of survivors at age x should be equal to the number of deaths in each year of age for all the following years.
- ${}_n d_x = \ell_x - \ell_{x+n} = \sum_{k=0}^{n-1} d_{x+k}$: the number of deaths within n years should be equal to the number of deaths in each year of age for the next n years.
- Finally, the probability that (x) survives the next n years but dies the following m years after that can be derived using:

$$ {}_{n|m}q_x = {}_n p_x - {}_{n+m}p_x = \frac{{}_m d_{x+n}}{\ell_x} = \frac{\ell_{x+n} - \ell_{x+n+m}}{\ell_x}. $$

The force of mortality

- It is easy to show that the force of mortality can be expressed in terms of life table function as:

$$\mu_x = -\frac{1}{\ell_x} \cdot \frac{d\ell_x}{dx}.$$

- Thus, in effect, we can also write

$$\ell_x = \ell_0 \cdot \exp\left(-\int_0^x \mu_z dz \right).$$

- With a simple change of variable, it is easy to see also that

$$\mu_{x+t} = -\frac{1}{\ell_{x+t}} \cdot \frac{d\ell_{x+t}}{dt} = -\frac{1}{{}_tp_x} \cdot \frac{d\,{}_tp_x}{dt}.$$

- It follows immediately that:

$$\frac{d}{dt}\,{}_tp_x = -\,{}_tp_x \mu_{x+t}.$$

Curtate expectation of life

- Analogously, the expected value of K_x, is called the curtate expectation of life defined by

$$\mathsf{E}[K_x] = e_x = \sum_{k=0}^{\infty} k \, {}_k p_x q_{x+k}.$$

- It can be shown (e.g. summation by parts) that

$$e_x = \sum_{k=1}^{\infty} {}_k p_x = \sum_{k=1}^{\infty} \frac{\ell_{x+k}}{\ell_x}.$$

- The temporary curtate expectation of life defined by

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_k p_x = \sum_{k=1}^{n} \frac{\ell_{x+k}}{\ell_x},$$

which gives the average number of completed years lived over the interval $(x, x+n]$ for a life (x).

Illustrative example 2

Suppose you are given the following extract from a life table:

x	ℓ_x
94	16,208
95	10,902
96	7,212
97	4,637
98	2,893
99	1,747
100	0

1. Calculate e_{95}.
2. Calculate the variance of K_{95}, the curtate future lifetime of (95).
3. Calculate $e_{95:\overline{3}|}$.

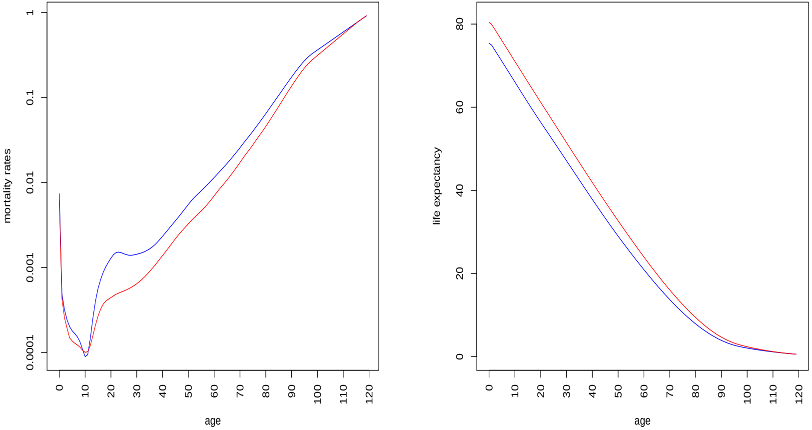

Figure : Source: Life Tables, 2007 from the Social Security Administration - male (blue), female (red)

Fractional age assumptions

- When adopting a life table (which may contain only integer ages), some assumptions are needed about the distribution between the integers.
- The two most common assumptions (or interpolations) used are (where $0 \leq t \leq 1$):
  1. linear interpolation (also called UDD assumption):

$$\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1}$$

  2. exponential interpolation (equivalent to constant force assumption):

$$\log \ell_{x+t} = (1-t) \log \ell_x + t \log \ell_{x+1}$$

Some results on the fractional age assumptions

Function	Linear (UDD)	Exponential (constant force)
${}_tq_x$	$t \cdot q_x$	$1-(1-q_x)^t$
μ_{x+t}	$\dfrac{q_x}{1-t\cdot q_x}$	$\mu = -\log p_x$
${}_tp_x\mu_{x+t}$	q_x	$\mu e^{-\mu t}$

Here we have $0 \leq t \leq 1$.

Illustrative example 3

You are given the following extract from a life table:

x	ℓ_x
55	85,916
56	84,772
57	83,507
58	82,114

Estimate ${}_{1.4}p_{55}$ and ${}_{0.5|1.6}q_{55}$ under each of the following assumptions for non-integral ages:

(a) UDD; and

(b) constant force.

Interpret these probabilities.

Fractional part of the year lived

- Denote by R_x the fractional part of a year lived in the year of death. Then we have

$$T_x = K_x + R_x$$

  where T_x is the time-until-death and K_x is the curtate future lifetime of (x).

- We can describe the joint probability distribution of (K_x, R_x) as

$$\mathsf{Pr}\left[(K_x = k) \cap (R_x \leq s)\right] = \mathsf{Pr}[k < T_x \leq k + s] = {}_k p_x \cdot {}_s q_{x+k},$$

  for $k = 0, 1, \ldots$ and for $0 < s < 1$.

- The UDD assumption is equivalent to the assumption that the fractional part R_x occurs uniformly during the year, i.e. $R_x \sim \mathsf{U}(0, 1)$.
  - It can be demonstrated that K_x and R_x are independent in this case.

Select and ultimate tables

- Group of lives underwritten for insurance coverage usually has different mortality than the general population (some test required before insurance is offered).
- Mortality then becomes a function of age $[x]$ at selection (e.g. policy issue, onset of disability) and duration t since selection.
- For select tables, notation such as ${}_tq_{[x]}$, ${}_tp_{[x]}$, and $\ell_{[x]+t}$, are then used.
- However, impact of selection diminishes after some time - the select period (denoted by r).
- In effect, we have

$$q_{[x]+j} = q_{x+j}, \text{ for } j \geq r.$$

Example of a select and ultimate table

$[x]$	$1000q_{[x]}$	$1000q_{[x]+1}$	$1000q_{x+2}$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
30	0.222	0.330	0.422	9,907	9,905	9,901	32
31	0.234	0.352	0.459	9,903	9,901	9,897	33
32	0.250	0.377	0.500	9,899	9,896	9,893	34
33	0.269	0.407	0.545	9,894	9,892	9,888	35
34	0.291	0.441	0.596	9,889	9,887	9,882	36

- From this table, try to compute probabilities such as:
  (a) ${}_2p_{[30]}$;
  (b) ${}_5p_{[30]}$;
  (c) ${}_{1|}q_{[31]}$; and
  (d) ${}_3q_{[31]+1}$.

Illustrative example 4

A select and ultimate table with a three-year select period begins at selection age x.

You are given the following information:

- $\ell_{x+6} = 90,000$
- $q_{[x]} = \frac{1}{6}$
- ${}_5p_{[x+1]} = \frac{4}{5}$
- ${}_3p_{[x]+1} = \frac{9}{10} \cdot {}_3p_{[x+1]}$.

Evaluate $\ell_{[x]}$.

Illustrative example 5

You are given the following extract from a select and ultimate life table:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
60	29,616	29,418	29,132	62
61	29,131	28,920	28,615	63
62	28,601	28,378	28,053	64

Calculate $1000 {}_{0.7}q_{[60]+0.8}$, assuming a constant force of mortality at fractional ages.

Illustrative example 6

You are given the following extract from a select and ultimate life table:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
65	80,625	79,954	78,839	67
66	79,137	78,402	77,252	68
67	77.575	76,770	75,578	69

Approximate $\mathring{e}_{[65]:\overline{2}|}$ using the trapezium (trapezoidal) rule with $h = 0.5$ and assuming UDD for fractional ages.

Illustrative example 7

The mortality pattern of a life (x) is based on a select and ultimate survival model where the ultimate part follows De Moivre's law with $\omega = 80$.

You are given:

$$q_{[x]+t} = \begin{cases} \frac{t+1}{t+2} q_{x+t}, & t = 0, 1, 2 \\ q_{x+t}, & t = 3, 4, \ldots \end{cases}$$

Calculate the probability that an individual, insured (or selected) one year ago at age 35, will die between age 38 and 40.

Illustrative example 8 - modified SOA MLC Spring 2012

Suppose you are given:

- $p_{50} = 0.98$
- $p_{51} = 0.96$
- $e_{51.5} = 22.4$
- The force of mortality is constant between ages 50 and 51.
- Deaths are uniformly distributed between ages 51 and 52.

Calculate $e_{50.5}$.

Illustrative example 9 - modified SOA MLC Spring 2012

In a 2-year select and ultimate mortality table, you are given:

- $q_{[x]+1} = 0.96\, q_{x+1}$
- $\ell_{65} = 82,358$
- $\ell_{66} = 81,284$

Calculate $\ell_{[64]+1}$.

Mortality projection factors

Read Section 3.11

Only other symbol used in the MLC exam

Expression	SOA will adopt
number of lives	l_x