Annuities

Lecture: Weeks 9-11

What are annuities?

An annuity is a series of payments that could vary according to:

- timing of payment

beginning of year (annuity-due)

end of year (annuity-immediate)

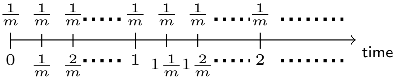
- with fixed maturity

n-year annuity-due

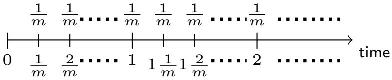
n-year annuity-immediate

- more frequently than once a year

annuity-due payable m-thly

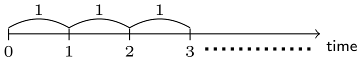


annuity-immediate payable m-thly



- payable continuously

continuous annuity



- varying benefits

Review of annuities-certain

annuity-due

$1 + v + v^2 + .. + v^{n-1}$

annuity-immediate

- payable annually

$$\ddot{a}_{\overline{n}|} = \sum_{k=0}^{n} v^{k-1} = \frac{1-v^n}{d}$$

$1 - v$

$$a_{\overline{n}|} = \sum_{k=1}^{n} v^k = \frac{1-v^n}{i}$$

$v + v^2 + .. + v^n$

- payable m times a year

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1}{m} \sum_{k=0}^{mn-1} v^{k/m} = \frac{1-v^n}{d^{(m)}}$$

$$a^{(m)}_{\overline{n}|} = \frac{1}{m} \sum_{k=1}^{mn} v^k = \frac{1-v^n}{i^{(m)}}$$

continuous annuity

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{1-v^n}{\delta}$$

Chapter summary

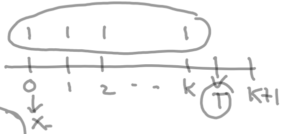
- Life annuities
  - series of benefits paid contingent upon survival of a given life
  - single life considered
  - actuarial present values (APV) or expected present values (EPV)
  - actuarial symbols and notation
- Types of annuities
  - discrete - due or immediate
    - payable more frequently than once a year
  - continuous
  - varying payments
- "Current payment techniques" APV formulas
- Chapter 5 of Dickson, et al.

Whole life annuity-due

forever until you die

- Pays a benefit of a unit $\$1$ at the beginning of each year that the annuitant (x) survives.

- The present value random variable is



$$Y = \ddot{a}_{\overline{K+1}|}$$

where K, in short for K_x, is the curtate future lifetime of (x).

- The actuarial present value of a whole life annuity-due is

Selection

$$\begin{aligned}
\ddot{a}_x &= \mathsf{E}[Y] = \mathsf{E}\left[\ddot{a}_{\overline{K+1}|}\right] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \Pr[K=k] \\
&= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \cdot {}_{k}p_x \, q_{x+k}
\end{aligned}$$

Current payment technique

Benefit = 1 always

By writing the PV random variable as

$$Y = I(T>0) + vI(T>1) + v^2 I(T>2) + \cdots = \sum_{k=0}^{\infty} v^k I(T>k),$$

one can immediately deduce that

$1 + v\,p_x + v^2\,{}_2p_x + \ldots$

$$\begin{aligned}
\ddot{a}_x &= \mathsf{E}[Y] = \mathsf{E}\left[\sum_{k=0}^{\infty} v^k I(T>k)\right] \\
&= \sum_{k=0}^{\infty} v^k \mathsf{E}[I(T>k)] = \sum_{k=0}^{\infty} v^k \mathsf{Pr}[\cancel{I}(T>k)] \\
&= \sum_{k=0}^{\infty} v^k\,{}_kp_x = \sum_{k=0}^{\infty} {}_kE_x = \sum_{k=0}^{\infty} A_{x:\overset{1}{\overline{k|}}}.
\end{aligned}$$

$B \cdot \ddot{a}_x$

$\ddot{a}_x = \sum_{k=0}^{\infty} v^k\,{}_kp_x$

Example.

A straightforward proof of $\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} \cdot {}_{k|}q_x = \sum_{k=0}^{\infty} v^k\,{}_kp_x$ is in **Exercise 5.1**.

$$\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} \underbrace{{}_k p_x \, q_{x+k}}$$

$${}_{k|}q_x = \left({}_k p_x - {}_{k+1} p_x\right)$$

$$\left(\frac{1-v^{k+1}}{d}\right) = \frac{1}{d}\left[\sum_{k=0}^{\infty}\left({}_k p_x - {}_{k+1}p_x\right) - \sum_{k=0}^{\infty} v^{k+1}\left({}_k p_x - {}_{k+1}p_x\right)\right]$$

$$1 + \cancel{{}_1 p_x} + \cancel{{}_2 p_x} + \cancel{{}_3 p_x} + \ldots$$
$$- \cancel{{}_1 p_x} - \cancel{{}_2 p_x} - \cancel{{}_3 p_x}$$

$$v \, {}_0 p_x + v^2 \, {}_1 p_x + v^3 \, {}_2 p_x + \ldots$$
$$- v \, {}_1 p_x - v^2 \, {}_2 p_x - \ldots$$

$$1 - \frac{1}{1+i} = i \cdot v$$

$$d = 1 - v$$
$$\downarrow$$
$$i \cdot v$$

$$= \frac{1}{d}\left[(1-v) + v(1-v)\,{}_1 p_x + v^2(1-v)\,{}_2 p_x + \ldots\right]$$

d, $1-v$ cancel

$$= \underbrace{1 + v \, p_x + v^2 \, {}_2 p_x + \ldots}_{\text{CPT}}$$

Current payment technique - continued

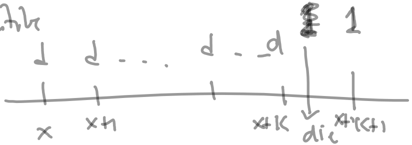
- The commonly used formula $\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x$ is the so-called current payment technique for evaluating life annuities. CPT
- Indeed, this formula gives us another intuitive interpretation of what life annuities are: they are nothing but sums of pure endowments (you get a benefit each time you survive).
- The primary difference lies in when you view the payments: one gives the series of payments made upon death, the other gives the payment made each time you survive.

$$A_x = 1 - d\,\ddot{a}_x$$

$$\cancel{1 = A_x}$$

$$1 = d\,\ddot{a}_x + A_x$$

intuitive



Some useful formulas

Life annuities to life insurance

By recalling that $\ddot{a}_{\overline{K+1}|} = \dfrac{1 - v^{K+1}}{d}$, we can use this to derive:

- relationship to whole life insurance

whole
Z = life insurance

$$\ddot{a}_x = \mathsf{E}\left[\frac{1 - v^{K+1}}{d}\right] = \frac{1}{d}\left(1 - A_x\right).$$

Alternatively, we write: $\boxed{A_x = 1 - d\ddot{a}_x.}$ very important formula!

- the variance formula

$$\mathsf{Var}[Y] = \mathsf{Var}\left[\ddot{a}_{\overline{K+1}|}\right] = \frac{1}{d^2}\mathsf{Var}\left[v^{K+1}\right] = \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right].$$

$\neq {}^2\ddot{a}_x - (\ddot{a}_x)^2$

Illustrative example 1

Suppose you are interested in valuing a whole life annuity-due issued to (95). You are given:

- $i = 5\%$, and
- the following extract from a life table:

x	95	96	97	98	99	100
ℓ_x	100	70	40	20	4	0

30 30 20 16 4

1. Express the present value random variable for a whole life annuity-due to (95). → $Y = \ddot{a}_{\overline{K_{95}+1|}} = \ddot{a}_{\overline{K+1|}}$ $K = K_{95}$
2. Calculate the expected value of this random variable.
3. Calculate the variance of this random variable.

$$E[Y] = \sum_{k=0}^{\infty} v^k \, {}_kp_{95} = \sum_{k=0}^{\infty} v^k \frac{l_{95+k}}{l_{95}} \qquad {}_kp_x = \frac{l_{x+k}}{l_x}$$

$$v = \frac{1}{1.05} \qquad = \frac{1}{l_{95}}\left[l_{95} + v\, l_{96} + v^2 l_{97} + \dots \right]$$

$$= \underbrace{\frac{1}{100}\left[100 + \frac{1}{1.05}\cdot 70 + \frac{1}{1.05^2}\cdot 40 + \frac{1}{1.05^3}\, 20 + \frac{1}{1.05^4}\cdot 4 \right]}_{2.235154}$$

$$Var[Y] = \frac{1}{d^2}\left[{}^2A_x - (A_x)^2 \right]$$

K	$P_r(K=k) = d_{x+k}/l_x$	$\ddot{a}_{\overline{K+1}|} = 1+v+v^2+\dots+v^K = Y$	$\ddot{a}_{\overline{K+1}|} P_r(K=k)$	$(\ddot{a}_{\overline{K+1}|})^2 P_r(K=k)$
0	30/100	1	⋮	⋮
1	30/100	$1+v = 1.952381$	⋮	
2	20/100	$1+v+v^2 = 2.859410$	⋮	.
3	16/100	$1+v+v^2+v^3$	⋮	
4	4/100	$1+v+v^2+v^3+v^4$		
			2.235154	6.123433

$$\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2$$

$$= 6.123433 - (2.235154)^2 = \underline{\underline{1.127508}}$$

$$d = 1 - v$$

$${}^2d = 1 - v^2$$

$$= \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$$

$$d = \frac{.05}{1.05}$$

$$A_x = 1 - d\,\ddot{a}_x$$

$$= 1 - \frac{.05}{1.05}(2.235154) = .8935641$$

$${}^2A_x = 1 - (1 - v^2)\,{}^2\ddot{a}_x$$

$$\delta \to 2\delta$$

$$v \to v^2$$

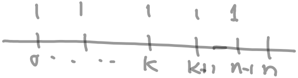
$${}^2\ddot{a}_{95} = 1 + v^2\,{}_1p_{95} + v^4\,{}_2p_x + v^6\,{}_3p_x + \ldots$$

$$= \underbrace{\frac{1}{\underset{100}{l_{95}}}\left[\underset{100}{l_{95}} + \frac{1}{1.05^2}\underset{70}{l_{96}} + \ldots\right.}_{= \underline{2.140318}}$$

$$= \underbrace{\frac{1}{\left(\frac{.05}{1.05}\right)^2}\left[\overbrace{1 - \left(1 - \frac{1}{1.05^2}\right)(2.140318)}^{{}^2A_x} - (.8935641)^2\right]}_{= \underline{1.127508}}$$

Temporary life annuity-due

- Pays a benefit of a unit $\$1$ at the beginning of each year so long as the annuitant (x) survives, for up to a total of n years, or n payments.
- The present value random variable is



$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|}, & K < n \\ \ddot{a}_{\overline{n}|}, & K \geq n \end{cases} = \ddot{a}_{\overline{\min(K+1,n)}|}.$$

- The APV of an n-year life annuity-due can be expressed as

$\ddot{a}_x$

$$\ddot{a}_{x:\overline{n}|} = \mathsf{E}[Y] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\, {}_kp_x\, q_{x+k} + \ddot{a}_{\overline{n}|}\, {}_np_x$$

using the current payment technique

$$\text{CPT} = \sum_{k=0}^{n-1} v^k\, {}_kp_x.$$

stops after n payments!

Some useful formulas

$\ddot{a}_{\overline{\min(K+1,n)}|} = \dfrac{1 - v^{\min(K+1,n)}}{d}$

Notice that $Z = v^{\min(K+1,n)}$ is the PV random variable associated with an n-year endowment insurance, with death benefit payable at EOY. Similar to the case of the whole life, we can use this to derive:

$= \begin{cases} v^{K+1}, & K < n \\ v^n, & K \geq n \end{cases}$

- relationship to whole life insurance

$1 = d\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}$

$$\ddot{a}_{x:\overline{n}|} = \mathsf{E}\left[\frac{1-Z}{d}\right] = \frac{1}{d}\left(1 - A_{x:\overline{n}|}\right).$$

Alternatively, we write: $\boxed{A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}.}$ very important formula!

n-year endowment n-year temporary

- the variance formula

$$\mathsf{Var}[Y] = \frac{1}{d^2}\mathsf{Var}[Z] = \frac{1}{d^2}\left[{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right].$$

Nov 17 Monday

Φ $1 \; 1 \; \ldots \; 1$

$$Y = \begin{cases} 0, & K < n \\ v^n \, \ddot{a}_{\overline{K+1-n}|}, & K \geq n \end{cases}$$

n-year deferred life annuity starts after n years

$$E[Y] = {}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^n \ddot{a}_{\overline{k+1-n}|} \, {}_{k|}q_x$$

$$= \sum_{k=n}^{\infty} v^k \, {}_{k}p_x$$

$$\boxed{\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x \, \ddot{a}_{x+n}}$$

Deferred whole life annuity-due

- Pays a benefit of a unit $\$1$ at the beginning of each year while the annuitant (x) survives from $x+n$ onward.
- The PV random variable can be expressed in a number of ways:

$$Y = \begin{cases} 0, & 0 \le K < n \\ {}_{n|}\ddot{a}_{\overline{K+1-n}|} = v^n \ddot{a}_{\overline{K+1-n}|} = \ddot{a}_{\overline{K+1}|} - \ddot{a}_{\overline{n}|}, & K \ge n \end{cases}.$$

- The APV of an n-year deferred whole life annuity can be expressed as

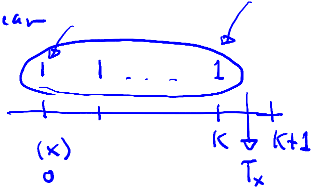
$$\begin{aligned} {}_{n|}\ddot{a}_x &= \mathsf{E}[Y] = \sum_{k=n}^{\infty} v^k \, {}_kp_x \\ &= {}_nE_x \, \ddot{a}_{x+n} = \ddot{a}_x - \ddot{a}_{x:\overline{n}|}. \end{aligned}$$

Life annuity-due

Contingent on survival

beginning of year

$B=1$



whole life

$$Y = PV = \ddot{a}_{\overline{K+1}|}$$

$$\text{APV(annuity)} = E[Y] = E[\ddot{a}_{\overline{K+1}|}] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\, \underbrace{{}_{k|}q_x}_{d_{x+k}/l_x}$$

$$\underbrace{1+v+v^2+\dots+v^k}_{\frac{1-v^{k+1}}{1-v} = \frac{1-v^{k+1}}{d}}$$

$$\frac{1-v^{K+1}}{d}$$

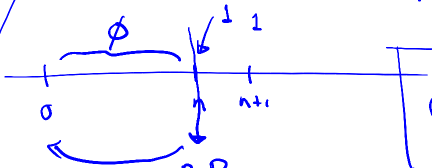
$$= \sum_{k=0}^{\infty} v^k\, {}_k p_x \quad \text{CPT}$$

$$Var[Y] = \frac{1}{d^2} Var(v^{K+1}) = \frac{1}{d^2}\left({}^2A_x - (A_x)^2\right)$$

$$\ddot{a}_x = \frac{1-A_x}{d} \quad \text{OR} \quad \boxed{A_x = 1 - d\,\ddot{a}_x}$$

other types: n-year temporary $\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_kp_x$

n-year deferred ${}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k \, {}_kp_x$



${}_nE_x = v^n \, {}_np_x$

$$\boxed{\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x}$$

${}_{n|}\ddot{a}_x = {}_nE_x \, \ddot{a}_{x+n}$

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|}, & K < n \\ \ddot{a}_{\overline{n}|}, & K \geq n \end{cases} = \ddot{a}_{\overline{\min(K+1,n)}|} = \frac{1 - v^{\min(K+1,n)}}{d}$$

$$E[Y] = \ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d} \iff \boxed{A_{x:\overline{n}|} = 1 - d\,\ddot{a}_{x:\overline{n}|}}$$

Variance of a deferred whole life annuity-due

To derive the variance is not straightforward. The best strategy is to work with

$$Y = \begin{cases} 0, & 0 \leq K < n \\ v^n \, \ddot{a}_{\overline{K+1-n|}}, & K \geq n \end{cases}$$

$\Rightarrow E[Y] = {}_{n|}\ddot{a}_x$

and use

$$\begin{aligned} \mathsf{Var}[Y] &= \mathsf{E}[Y^2] - (\mathsf{E}[Y])^2 \\ &= \sum_{k=n}^{\infty} v^{2n} \left(\ddot{a}_{\overline{k+1-n|}} \right)^2 {}_{k|}q_x - \left({}_{n|}\ddot{a}_x \right)^2 \end{aligned}$$

Apply a change of variable of summation to say $k^* = k - n$ and then the variance of a whole life insurance issued to $(x+n)$.

The variance of Y finally can be expressed as

$$\mathsf{Var}[Y] = \frac{2}{d} v^{2n} {}_n p_x \left(\ddot{a}_{x+n} - {}^2\ddot{a}_{x+n} \right) + {}_{n|}^{2}\ddot{a}_x - \left({}_{n|}\ddot{a}_x \right)^2.$$

Illustrative example 2

from first

Suppose you are interested in valuing a 2-year deferred whole life annuity-due issued to (95). You are given:

1 1 1 0

(95) 2 97 98 99 100

- $i = 6\%$ and
- the following extract from a life table:

250 350 175 150 75

x	95	96	97	98	99	100
ℓ_x	1000	750	400	225	75	0

1. Express the present value random variable for this annuity.
2. Calculate the expected value of this random variable.
3. Calculate the variance of this random variable.

① PV random variable $Y = \begin{cases} \phi, & K < 2 \\ v^2 \ddot{a}_{\overline{K+1-2}|}, & K \geq 2 \end{cases}$ $v = \frac{1}{1.06}$

$_{2|}\ddot{a}_{\overline{K+1}|}$ should be $_{2|}\ddot{a}_{\overline{K+1-2}|}$

② $E[Y] = {}_{2|}\ddot{a}_{95} = \sum_{k=2}^{\infty} v^k {}_kp_{95} = \dfrac{v^2 l_{97} + v^3 l_{98} + v^4 l_{99}}{l_{95}}$ (400, 225, 75; l_{95} = 1000)

$= .6043199$

$= \frac{d_{x+k}}{l_x}$

K	$P_r(K=k)$	$Y = PV$	$Y \cdot P_r(K=k)$	$Y^2 P_r(K=k)$
0	.250	ϕ	ϕ	ϕ
1	.350	ϕ	ϕ	ϕ
2	.175	v^2	.8899964 (.175)	$(.8899964)^2(.175)$
3	.150	v^2+v^3	1.7296157 (.150)	$(1.7296157)^2(.150)$
4	.075	$v^2+v^3+v^4$	2.5217094 (.075)	⋮
	$\Sigma = 1$		$\Sigma = .6043199$	$\Sigma = 1.064278$

$$Var[Y] = 1.064278 - (.6043199)^2$$

$$= .6990758$$

Insurance: $A_x = v q_x + v p_x A_{x+1}$

Annuities: $\ddot{a}_x = 1 + v p_x \ddot{a}_{x+1}$

$$= \underbrace{1 + v p_x + v^2 {}_2p_x \ddot{a}_{x+2}}$$

sketch: $\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x = 1 + \underbrace{\sum_{k=1}^{\infty} v^k {}_kp_x}_{v p_x \ddot{a}_{x+1}}$ $\quad s = k-1$, $k = s+1$

Recursive relationships

- The following relationships are easy to show:

$$\begin{aligned}\ddot{a}_x &= 1 + vp_x\ddot{a}_{x+1} = 1 + {}_1E_x\ddot{a}_{x+1} \\ &= 1 + vp_x + v^2{}_2p_x\ddot{a}_{x+2} = 1 + {}_1E_x + {}_2E_x\ddot{a}_{x+2}\end{aligned}$$

- In general, because E's are multiplicative, we can generalized this recursions to

$v^k{}_kp_x = {}_kE_x$

$$\ddot{a}_x = \sum_{k=0}^{\infty} {}_kE_x = \sum_{k=0}^{n-1} {}_kE_x + \sum_{k=n}^{\infty} {}_kE_x$$

sums of pure endowment

apply change of variable $k^* = k - n$

$$\begin{aligned} &= \ddot{a}_{x:\overline{n}|} + \sum_{k^*=0}^{\infty} {}_nE_x\ {}_{k^*}E_{x+n} = \ddot{a}_{x:\overline{n}|} + {}_nE_x \sum_{k^*=0}^{\infty} {}_{k^*}E_{x+n} \\ &= \ddot{a}_{x:\overline{n}|} + {}_nE_x\ \ddot{a}_{x+n} = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x\end{aligned}$$

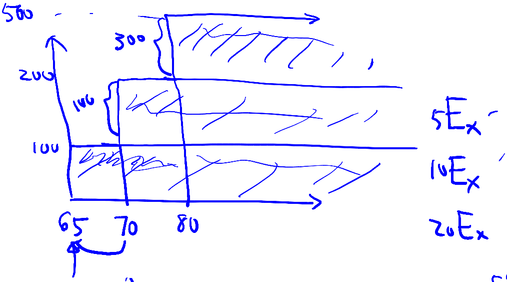
- The last term shows that a whole life annuity is the sum of a term life annuity and a deferred life annuity.

Varying due

$i = 6\%$

Table is ILT

Illustrative Life Table



Calculate APV of this annuity.

$$\text{APV(annuity)} = 100\,\underbrace{\ddot{a}_{65}}_{9.8969} + 100\,\underbrace{{}_5E_{65}}_{.65623}\,\overbrace{\ddot{a}_{70}}^{8.5693} + 300\,{}_{15}E_{65}\,\overbrace{\ddot{a}_{80}}^{5.9050}$$

$${}_{15}E_{65} = \underbrace{{}_5E_{65}}_{.65623}\,{}_{10}E_{70} = {}_{10}E_{65}\,{}_5E_{75} = .33037$$

$$= 1{,}936.092$$

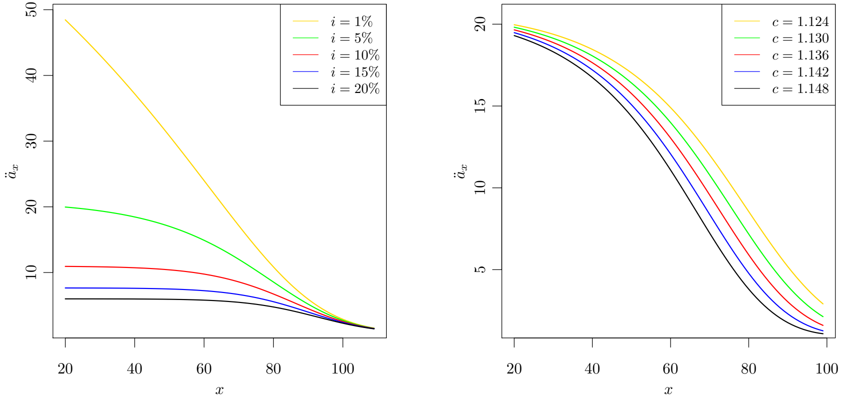


Figure : Comparing APV of a whole life annuity-due for based on the Standard Ultimate Survival Model (Makeham with $A = 0.00022$, $B = 2.7 \times 10^{-6}$, $c = 1.124$). Left figure: varying i. Right figure: varying c with $i = 5\%$

Whole life annuity-immediate

- Procedures and principles for annuity-due can be adapted for annuity-immediate.
- Consider the whole life annuity-immediate, the PV random variable is clearly $Y = a_{\overline{K|}}$ so that APV is given by

$$a_x = \mathsf{E}[Y] = \sum_{k=0}^{\infty} a_{\overline{K|}} \cdot {}_kp_x \, q_{x+k} = \sum_{k=1}^{\infty} v^k \, {}_kp_x.$$

- Relationship to life insurance:

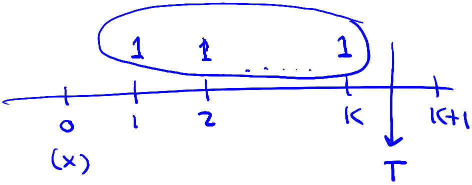
$$Y = \frac{1}{i}\left(1 - v^K\right) = \frac{1}{i}\left[1 - (1+i)v^{K+1}\right]$$

leads to $1 = ia_x + (1+i)A_x$.

- Interpretation of this equation - to be discussed in class.

Whole life annuity - immediate

$$\ddot{a}_x - a_x = 1$$



$$Y = a_{\overline{K}|}$$

$$a_x = E[Y] = \sum_{k=0}^{\infty} a_{\overline{k}|}\ {}_{k|}q_x = \cdots\cdots = \sum_{k=1}^{\infty} v^k\ {}_kp_x \quad \text{CPT}$$

$$a_{\overline{k}|} = \underbrace{v + v^2 + \cdots + v^k}_{\frac{v(1-v^k)}{1-v} = \frac{1-v^k}{i}}$$

$$Y = a_{\overline{K}|} = \frac{1-v^K}{i} = \frac{1-v^{K+1}(1+i)}{i}$$

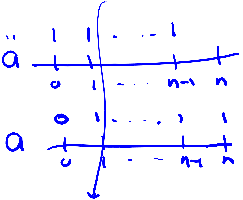
$$\mathrm{Var}[Y] = \frac{(1+i)^2}{i^2}\left[{}^2A_x - (A_x)^2\right]$$

$$= \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$$

$$\frac{1+i}{i} = \frac{1}{i \cdot v} = \frac{1}{d}$$

$\ddot{a}_x = 1 + a_x$ ✓

$\ddot{a}_{x:\overline{n}|} = 1 + \cancel{a_{x:\overline{n}|}}$ $\quad a_{x:\overline{n-1}|}$



$= 1 + a_{x:\overline{n}|} - v^n \, {}_np_x$

${}_{n|}a_x$ vs ${}_{n|}\ddot{a}_x$

$$Y = \frac{1-v^K}{i} = \frac{1 - v^{K+1}(1+i)}{i} \qquad v(1+i) = 1$$

$$a_x = \frac{1 - A_x(1+i)}{i}$$

$$i a_x = 1 - A_x(1+i)$$

$$1 = i a_x + (1+i) A_x$$ ✓

vs

$$1 = d\ddot{a}_x + A_x$$ ✓

$$a_x = v p_x + v p_x a_{x+1}$$

n-year temprary: $a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \, {}_kp_x$

n-year deferred: ${}_{n|}a_x = \sum_{k=n+1}^{\infty} v^k \, {}_kp_x$

$+ \Rightarrow a_x = a_{x:\overline{n}|} + {}_{n|}a_x$

${}_{n|}a_x = {}_nE_x \, a_{x+n}$

Other types of life annuity-immediate

- For an n-year life annuity-immediate:
  - Find expression for the present value random variable.
  - Express formulas for its actuarial present value or expectation.
  - Find expression for the variance of the present value random variable.
- For an n-year deferred whole life annuity-immediate:
  - Find expression for the present value random variable.
  - Give expressions for the actuarial present value.
- Details to be discussed in lecture.

Exam ends here

Reminder: Monday Nov 17

Illustrative Life Table

~~com~~ calculator

2 ~~sheet~~ cheat sheets

Review: ~~Wednesday~~

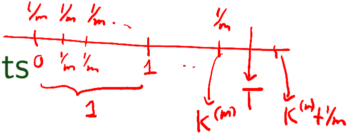
Thursday, Nov 13

C-304 3-5pm

Wells Problems

due, immediate

Life annuities with m-thly payments



In practice, life annuities are often payable more frequently than once a year, e.g. monthly ($m = 12$), quarterly ($m = 4$), or semi-annually ($m = 2$).

Here, we define the random variable $K_x^{(m)}$, or simply $K^{(m)}$, to be the complete future lifetime rounded down to the nearest $1/m$-th of a year.

For example, if the observed $T = 45.86$ for a life (x) and $m = 4$, then the observed $K^{(4)}$ is $45\frac{3}{4}$.

Indeed, we can write

$$K^{(m)} = \frac{1}{m} \lfloor mT \rfloor ,$$

where $\lfloor \, \rfloor$ is greatest integer (or floor) function.

whole life annuity-due payable m times a year of \$1 a year

1/m each payment

$$Y = \ddot{a}_{\overline{K^{(m)}+1/m}|} = \frac{1-V^{K^{(m)}+1/m}}{d^{(m)}}$$

$$\underbrace{\ddot{a}_{\overline{K^{(m)}+1/m}|}}_{\frac{1}{m}\left(1+v^{1/m}+v^{2/m}+\cdots v^{K^{(m)}}\right)}$$

$e^{\delta}=1+i$

$=\left(1+\frac{i^{(m)}}{m}\right)^{m}$

$=\left(1-\frac{d^{(m)}}{m}\right)^{-m}$

$=1-d$

$$\text{APV(annuity)} = E[Y] = \ddot{a}_x^{(m)} = \cdots = \underbrace{\sum_{k=0}^{\infty} \frac{1}{m} v^{k/m} {}_{\frac{k}{m}}p_x}_{\frac{1}{m}+\frac{1}{m}v^{\frac{1}{m}}{}_{\frac{1}{m}}p_x+v^{\frac{2}{m}}{}_{\frac{2}{m}}p_x+\cdots}$$

$$\ddot{a}_x^{(m)} = \frac{1-A_x^{(m)}}{d^{(m)}} \iff A_x^{(m)} = 1-d^{(m)}\ddot{a}_x^{(m)}$$

$$\text{Var}[Y] = \left(\frac{1}{d^{(m)}}\right)^2 \left[\underbrace{{}^2A_x^{(m)}}_{2\delta} - \left(A_x^{(m)}\right)^2 \right]$$

n-year life annuity: $APV = \ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_x - {}_{n|}\ddot{a}^{(m)}_x$

n-year deferred: $APV = {}_{n|}\ddot{a}^{(m)}_x = {}_nE_x \, \ddot{a}^{(m)}_{x+n}$

Whole life annuity-due payable m times a year

- Consider a whole life annuity-due with payments made m times a year. Its PV random variable can be expressed as

$$Y = \ddot{a}^{(m)}_{\overline{K^{(m)}+(1/m)}|} = \frac{1 - v^{K^{(m)}+(1/m)}}{d^{(m)}}.$$

- The APV of this annuity is

CPT

$$\mathsf{E}[Y] = \ddot{a}^{(m)}_x = \frac{1}{m}\sum_{h=0}^{\infty} v^{h/m} \cdot {}_{h/m}p_x = \frac{1 - A^{(m)}_x}{d^{(m)}}.$$

- Variance is

$$\mathsf{Var}[Y] = \frac{\mathsf{Var}\left[v^{K^{(m)}+(1/m)}\right]}{\left(d^{(m)}\right)^2} = \frac{{}^2A^{(m)}_x - \left(A^{(m)}_x\right)^2}{\left(d^{(m)}\right)^2}.$$

Some useful relationships

Here we list some important relationships regarding the life annuity-due with m-thly payments (Note - these are exact formulas):

- $1 = d\ddot{a}_x + A_x = d^{(m)}\ddot{a}_x^{(m)} + A_x^{(m)}$

leave it as an exercise

- $\ddot{a}_x^{(m)} = \dfrac{d}{d^{(m)}}\ddot{a}_x - \dfrac{1}{d^{(m)}}\left(A_x^{(m)} - A_x\right) = \ddot{a}_{\overline{1|}}^{(m)}\ddot{a}_x - \ddot{a}_{\overline{\infty|}}^{(m)}\left(A_x^{(m)} - A_x\right)$

- $\ddot{a}_x^{(m)} = \dfrac{1 - A_x^{(m)}}{d^{(m)}} = \ddot{a}_{\overline{\infty|}}^{(m)} - \ddot{a}_{\overline{\infty|}}^{(m)} A_x^{(m)}$

approx $\ddot{a}^{(m)} \to \ddot{a}$

$$\boxed{\sum_{k=0}^{\infty} \frac{1}{m} v^{k/m}\, {}_{k/m}p_x}$$

Approximation

① UDD

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x$$

$$\ddot{a}_x^{(m)} = $$

$$A_x^{(m)} = 1 - d^{(m)} \ddot{a}_x^{(m)}$$

$$= \frac{1 - A_x^{(m)}}{d^{(m)}} = \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}}$$

$$= \frac{i^{(m)} - i A_x}{i^{(m)} d^{(m)}} \qquad A_x \to (1 - d\ddot{a}_x)$$

$$= \underbrace{\frac{id}{i^{(m)} d^{(m)}}} \ddot{a}_x - \frac{(-i^{(m)} + i)}{i^{(m)} d^{(m)}}$$

$$= \underline{\underline{\alpha(m)\, \ddot{a}_x - \beta(m)}}$$

② Woolhouse!

Other types of life annuity-due payable m-thly

n-year term	PV random variable	Y	$= \ddot{a}^{(m)}_{\overline{\min(K^{(m)}+(1/m),n)}\rceil}$
	APV symbol	$\mathsf{E}[Y]$	$= \ddot{a}^{(m)}_{x:\overline{n}\rceil}$
	current payment technique		$= \dfrac{1}{m}\sum_{h=0}^{mn-1} v^{h/m} \cdot {}_{h/m}p_x$
	other relationships		$= \ddot{a}^{(m)}_x - {}_nE_x\, \ddot{a}^{(m)}_{x+n}$
	relation to life insurance		$= \dfrac{1}{d^{(m)}}\left[1 - A^{(m)}_{x:\overline{n}\rceil}\right]$
n-year deferred	PV random variable	Y	$= v^n \ddot{a}^{(m)}_{\overline{K^{(m)}+(1/m)-n}\rceil} I(K \geq n)$
	APV symbol	$\mathsf{E}[Y]$	$= {}_{n\vert}\ddot{a}^{(m)}_x$
	current payment technique		$= \dfrac{1}{m}\sum_{h=mn}^{\infty} v^{h/m} \cdot {}_{h/m}p_x$
	other relationships		$= {}_nE_x\, \ddot{a}^{(m)}_{x+n} = \ddot{a}^{(m)}_x - \ddot{a}^{(m)}_{x:\overline{n}\rceil}$
	relation to life insurance		$= \dfrac{1}{d^{(m)}}\left[{}_nE_x - {}_{n\vert}A^{(m)}_x\right]$

$$\ddot{a}_x^{(m)} = \alpha(m)\,\ddot{a}_x - \beta(m)$$

n-year temporary life

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - \underbrace{{}_{n|}\ddot{a}_x^{(m)}}_{{}_nE_x\,\ddot{a}_{x+n}^{(m)}}$$

$\ddot{a}_x^{(m)} \to \alpha(m)\,\ddot{a}_x - \beta(m)$

$\ddot{a}_{x+n}^{(m)} \to \alpha(m)\,\ddot{a}_{x+n} - \beta(m)$

$$= \alpha(m)\Big[\underbrace{\ddot{a}_x - {}_nE_x\,\ddot{a}_{x+n}}_{\ddot{a}_{x:\overline{n}|}}\Big] - \beta(m)\big[1 - {}_nE_x\big]$$

$$= \alpha(m)\,\ddot{a}_{x:\overline{n}|} - \beta(m)\big[1 - {}_nE_x\big]$$

n-year deferred: $\quad {}_{n|}\ddot{a}_x^{(m)} = \alpha(m)\,{}_{n|}\ddot{a}_x - \beta(m)\,{}_nE_x$

Illustrative example 3

Professor Balducci is currently age 60 and will retire immediately. He purchased a whole life annuity-due contract which will pay him on a monthly basis the following benefits:

- \$12,000 each year for the next 10 years;
- \$24,000 each year for the following 5 years after that; and finally,
- \$48,000 each year thereafter.

$\ddot{a}_x^{(12)} = 1 - d^{(12)} A_x^{(12)}$

You are given:

- $i = 3\%$ and the following table:

x	$1000 A_x^{(12)}$	${}_5p_x$
60	661.11	0.8504
65	712.33	0.7926
70	760.65	0.7164
75	804.93	0.6196

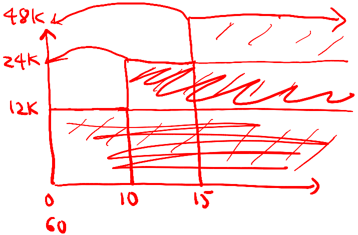
Calculate the APV of Professor Balducci's life annuity benefits.

$\frac{1-A_{70}^{(12)}}{d^{(12)}}$

APV(annuity) = ✓

$$12000\,\ddot{a}_{60}^{(12)} + 12000\,{}_{10}E_{60}\,\ddot{a}_{70}^{(12)}$$

$$+\ 24000\,{}_{15}E_{60}\,\ddot{a}_{75}^{(12)}$$ ✓



$1+.03 = \left(1-\frac{d^{(12)}}{12}\right)^{-12}$

$d^{(12)} = \left[1-(1.03)^{-1/12}\right]12$

$\ddot{a}_{60}^{(12)} = \left(1 - A_{60}^{(12)}\right)/d^{(12)}$

$A_{60}^{(12)} = .66111$

$d^{(12)} = .02952243$

$${}_{10}E_{60} = v^{10}\,{}_5p_{60}\,{}_5p_{65}$$

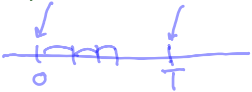
$$= \left(\frac{1}{1.03}\right)^{10} .8504\ \ .7926$$

$${}_{15}E_{60} = {}_{10}E_{60}\cdot{}_5E_{70} =$$

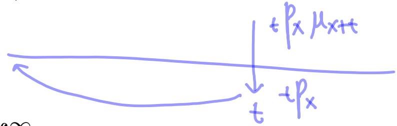
$${}_5E_{70} = v^5\,{}_5p_{70}$$

$$= \underline{235{,}693.10}$$ ✓

(Continuous) whole life annuity



- A life annuity payable continuously at the rate of one unit per year.
- One can think of it as life annuity payable m-thly per year, with $m \to \infty$.
- The PV random variable is $Y = \bar{a}_{\overline{T}|}$ where T is the future lifetime of (x).
- The APV of the annuity:



$\lim_{m\to\infty} \ddot{a}_x^{(m)}$

$\lim_{m\to\infty} a_x^{(m)}$

$$\bar{a}_x = \mathsf{E}[Y] = \mathsf{E}\left[\bar{a}_{\overline{T}|}\right] = \int_0^\infty \bar{a}_{\overline{t}|} \cdot {}_tp_x \mu_{x+t} dt$$

use integration by parts - see page 117 for proof

$$= \int_0^\infty \underbrace{v^t {}_tp_x}_{{}_tE_x} dt = \int_0^\infty {}_tE_x dt$$

$\ddot{a}_x = \sum_{t=0}^{\infty} v^t {}_tp_x$

$$\bar{a}_{\overline{T}|} = \frac{1 - v^T}{\delta}$$

$$\bar{a}_x = E\left[\bar{a}_{\overline{T}|}\right] = \frac{1 - \overbrace{E(v^T)}^{\bar{A}_x}}{\delta} \iff \bar{A}_x = 1 - \delta\,\bar{a}_x$$

vs

$$A_x = 1 - d\,\ddot{a}_x$$

$$\mathrm{Var}\left[\bar{a}_{\overline{T}|}\right] = \mathrm{Var}\left[\frac{1 - v^T}{\delta}\right]$$

$$= \frac{1}{\delta^2}\,\mathrm{Var}\left[v^T\right] = \frac{1}{\delta^2}\left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right]$$

$${}^2\bar{A}_x - \left(\bar{A}_x\right)^2$$

vs

$$\frac{1}{d^2}\left[{}^2A_x - \left(A_x\right)^2\right]$$

- continued

- One can also write expressions for the cdf and pdf of Y in terms of the cdf and pdf of T. For example,

$$\Pr[Y \le y] = \Pr\left[1 - v^T \le \delta y\right] = \Pr\left[T \le \frac{\log(1-\delta y)}{\log v}\right]$$

- Recursive relation: $\bar{a}_x = \bar{a}_{x:\overline{1|}} + v p_x \bar{a}_{x+1}$

- Variance expression: $\mathrm{Var}\left[\bar{a}_{\overline{T|}}\right] = \mathrm{Var}\left[\dfrac{1-v^T}{\delta}\right] = \dfrac{1}{\delta^2}\left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right]$

- Relationship to whole life insurance: $\bar{A}_x = 1 - \delta \bar{a}_x$

- Try writing explicit expressions for the APV and variance where we have constant force of mortality and constant force of interest.

deferred: $\;{}_{n|}\bar{a}_x = {}_nE_x \, \bar{a}_{x+n}$

temporary: $\;\bar{a}_{x:\overline{n|}} = \bar{a}_x - {}_{n|}\bar{a}_x$

$$= \bar{a}_x - {}_nE_x \, \bar{a}_{x+n}$$

$$\bar{A}_{x:\overline{n|}} = 1 - \delta \, \bar{a}_{x:\overline{n|}}$$

vs

$$\bar{A}_x = 1 - \delta \, \bar{a}_x$$

Temporary life annuity

- A (continuous) n-year temporary life annuity pays 1 per year continuously while (x) survives during the next n years.
- The PV random variable is $Y = \begin{cases} \bar{a}_{\overline{T}|}, & 0 \le T < n \\ \bar{a}_{\overline{n}|}, & T \ge n \end{cases} = \bar{a}_{\overline{\min(T,n)}|}$
- The APV of the annuity:

$\bar{a}_x = \underbrace{\bar{a}_{x:\overline{1}|}} + v p_x \bar{a}_{x+1}$

$$\begin{aligned} \bar{a}_{x:\overline{n}|} &= \mathsf{E}[Y] = \int_0^n \bar{a}_{\overline{t}|} \cdot {}_tp_x \mu_{x+t} dt \\ &+ \int_n^\infty \bar{a}_{\overline{n}|} \cdot {}_tp_x \mu_{x+t} dt = \int_0^n v^t \, {}_tp_x dt. \end{aligned}$$

- Recursive formula: $\bar{a}_{x:\overline{n}|} = \bar{a}_{x:\overline{1}|} + v p_x \bar{a}_{x+1:\overline{n-1}|}$.
- To derive variance, one way to get explicit form is to note that $Y = (1 - Z)/\delta$ where Z is the PV r.v. for an n-year endowment ins. [details in class.]

Deferred whole life annuity

- Pays a benefit of a unit $1 each year continuously while the annuitant (x) survives from $x+n$ onward.
- The PV random variable is

$$Y = \begin{cases} 0, & 0 \le T < n \\ v^n \, \bar{a}_{\overline{T-n}|}, & T \ge n \end{cases} = \begin{cases} 0, & 0 \le T < n \\ \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|}, & T \ge n \end{cases}.$$

- The APV [expected value of Y] of the annuity is

$${}_{n|}\bar{a}_x = {}_nE_x \, \bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n}|} = \int_n^\infty v^t \, {}_tp_x dt.$$

- The variance of Y is given by

$$\mathrm{Var}[Y] = \frac{2}{\delta} v^{2n} \, {}_np_x \left(\bar{a}_{x+n} - {}^2\bar{a}_{x+n} \right) - \left({}_{n|}\bar{a}_x \right)^2$$

Special mortality laws

- Just as in the case of life insurance valuation, we can derive nice explicit forms for "life annuity" formulas in the case where mortality follows:
  - constant force (or Exponential distribution); or
  - De Moivre's law (or Uniform distribution).
- Try deriving some of these formulas. You can approach them in a couple of ways:
  - Know the results for the "life insurance" case, and then use the relationships between annuities and insurances.
  - You can always derive it from first principles, usually working with the current payment technique.
- In the continuous case, one can use numerical approximations to evaluate the integral:
  - trapezium (trapezoidal) rule
  - repeated Simpson's rule

Life annuities with varying benefits

- Some of these are discussed in details in Section 5.10.
- You may try to remember the special symbols used, especially if the variation is a fixed unit of \$1 (either increasing or decreasing).
- The most important thing to remember is to apply similar concept of "discounting with life" taught in the life insurance case (note: this works only for valuing actuarial present values):
  - work with drawing the benefit payments as a function of time; and
  - use then your intuition to derive the desired results.

Methods for evaluating annuity functions

- Section 5.11
- Recursions:
    - For example, in the case of a whole life annuity-due on (x), recall $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$. Given a set of mortality assumptions, start with

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x$$

      and then use the recursion to evaluate values for subsequent ages.
- UDD: deaths are uniformly distribution between integral ages.
- Woolhouse's approximations

Uniform Distribution of Deaths (UDD)

Under the UDD assumption, we have derived in the previous chapter that the following holds:

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x$$

Then use the relationship between annuities and insurance:

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}$$

This leads us to the following result when UDD holds:

$$\ddot{a}_x^{(m)} = \alpha(m)\, \ddot{a}_x - \beta(m),$$

where

$$\begin{aligned} \alpha(m) &= s_{\overline{1}|}^{(m)} \ddot{a}_{\overline{1}|}^{(m)} = \frac{i}{i^{(m)}} \cdot \frac{d}{d^{(m)}} \\ \beta(m) &= \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} = \frac{i - i^{(m)}}{i^{(m)} d^{(m)}} \end{aligned}$$

Woolhouse's approximate fomulas

The Woolhouse's approximate formulas for evaluating annuities are based on the Euler-Maclaurin formula for numerical integration:

$$\int_0^\infty g(t)dt = h\sum_{k=0}^{\infty} g(kh) - \frac{h}{2}g(0) + \frac{h^2}{12}g'(0) - \frac{h^4}{720}g''(0) + \cdots$$

for some positive constant h. This formula is then applied to $g(t) = v^t\,{}_tp_x$ which leads us to

$$g'(t) = -v^t\,{}_tp_x\left(\delta - \mu_{x+t}\right).$$

We can obtain the following Woolhouse's approximate formula:

$\ddot{a}_x^{(12)} = \ddot{a}_x - \frac{11}{24}$

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$$

$\ddot{a}_x^{(12)} = \ddot{a}_x - \frac{11}{24} - \frac{12^2-1}{12(12)^2}(\delta+\mu_x)$

W2 W3

Approximating an n-year temporary life annuity-due with m-thly payments

Apply the Woolhouse's approximate formula to

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x} - {}_nE_x \, \ddot{a}^{(m)}_{x+n}$$

This leads us to the following Woolhouse's approximate formulas:

Use 2 terms (W2)	$\ddot{a}^{(m)}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - {}_nE_x\right)$
Use 3 terms (W3)	$\ddot{a}^{(m)}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - {}_nE_x\right)$
	$-\frac{m^2-1}{12m^2}\left[\delta + \mu_x - {}_nE_x\left(\delta + \mu_{x+n}\right)\right]$
Use 3 terms (W3*)	use approximation for force of mortality
(modified)	$\mu_x \approx -\frac{1}{2}\left[\log(p_{x-1}) + \log(p_x)\right]$

Numerical illustrations

We compare the various approximations: UDD, W2, W3, W3* based on the Standard Ultimate Survival Model with Makeham's law

$$\mu_x = A + Bc^x,$$

where $A = 0.00022$, $B = 2.7 \times 10^{-6}$ and $c = 1.124$.

The results for comparing the values for:

- $\ddot{a}^{(12)}_{x:\overline{10}|}$ with $i = 10\%$
- $\ddot{a}^{(2)}_{x:\overline{25}|}$ with $i = 5\%$

are summarized in the following slides.

Values of $\ddot{a}^{(12)}_{x:\overline{10}|}$ with $i = 10\%$

x	$\ddot{a}_x$	$\ddot{a}^{(12)}_x$	${}_{10}E_x$	Exact	UDD	W2	W3	W3*
20	10.9315	10.4653	0.384492	6.4655	6.4655	6.4704	6.4655	6.4655
30	10.8690	10.4027	0.384039	6.4630	6.4630	6.4679	6.4630	6.4630
40	10.7249	10.2586	0.382586	6.4550	6.4550	6.4599	6.4550	6.4550
50	10.4081	9.9418	0.377947	6.4295	6.4294	6.4344	6.4295	6.4295
60	9.7594	9.2929	0.363394	6.3485	6.3482	6.3535	6.3485	6.3485
70	8.5697	8.1027	0.320250	6.0991	6.0982	6.1044	6.0990	6.0990
80	6.7253	6.2565	0.213219	5.4003	5.3989	5.4073	5.4003	5.4003
90	4.4901	4.0155	0.057574	3.8975	3.8997	3.9117	3.8975	3.8975
100	2.5433	2.0505	0.000851	2.0497	2.0699	2.0842	2.0497	2.0496

Values of $\ddot{a}^{(2)}_{x:\overline{25}|}$ with $i = 5\%$

✓

x	$\ddot{a}_x$	$\ddot{a}_x^{(2)}$	${}_{25}E_x$	Exact	UDD	W2	W3	W3*
20	19.9664	19.7133	0.292450	14.5770	14.5770	14.5792	14.5770	14.5770
30	19.3834	19.1303	0.289733	14.5506	14.5505	14.5527	14.5506	14.5506
40	18.4578	18.2047	0.281157	14.4663	14.4662	14.4684	14.4663	14.4663
50	17.0245	16.7714	0.255242	14.2028	14.2024	14.2048	14.2028	14.2028
60	14.9041	14.6508	0.186974	13.4275	13.4265	13.4295	13.4275	13.4275
70	12.0083	11.7546	0.068663	11.5117	11.5104	11.5144	11.5117	11.5117
80	8.5484	8.2934	0.002732	8.2889	8.2889	8.2938	8.2889	8.2889
90	5.1835	4.9242	0.000000	4.9242	4.9281	4.9335	4.9242	4.9242
100	2.7156	2.4425	0.000000	2.4425	2.4599	2.4656	2.4424	2.4424

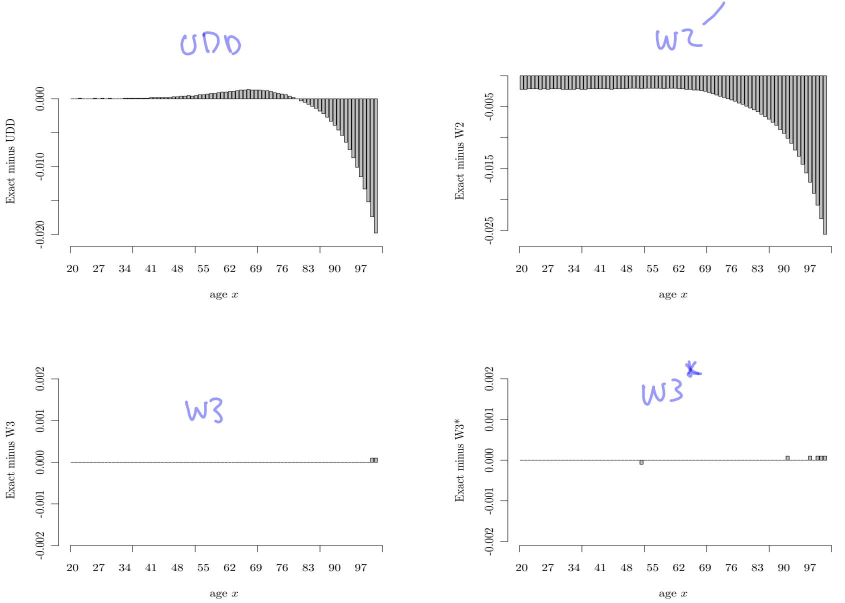


Figure : Visualizing the different approximations for $\ddot{a}^{(2)}_{x:\overline{25}|}$

Approximations

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x} - {}_nE_x \, \ddot{a}^{(m)}_{x+n}$$

UDD: $$\ddot{a}^{(m)}_{x} = \underbrace{\alpha(m)}_{\frac{i}{i^{(m)}}\frac{d}{d^{(m)}}} \ddot{a}_x - \underbrace{\beta(m)}_{\frac{i-i^{(m)}}{i^{(m)}d^{(m)}}}$$

Woolhouse: $$\ddot{a}^{(m)}_{x} = \underbrace{\underbrace{\ddot{a}_x - \frac{m-1}{2m}}_{W2 \quad 2 \text{ terms}} - \frac{m^2-1}{12m^2}(\delta + \mu_x)}_{W3 \quad 3 \text{ terms}}$$

$W3^*$ replace $\mu_x = -\frac{1}{2}\left[\log(p_{x-1}) + \log(p_x)\right]$

$m \to \infty \Rightarrow$ benefits continuous

Illustrative example 4

You are given:

- $i = 5\%$ and the following table:

x	ℓ_x	μ_x
49	811	0.0213
50	793	0.0235
51	773	0.0258
52	753	0.0284
53	731	0.0312
54	707	0.0344

Approximate $\ddot{a}^{(12)}_{50:\overline{3|}}$ based on the following methods:

1. UDD assumptions
2. Woolhouse's formula using the first two terms only
3. Woolhouse's formula using all three terms
4. Woolhouse's formula using all three terms but approximating the force of mortality

$$\ddot{a}_{50:\overline{3}|} = 1 + \underbrace{v}_{\frac{1}{1.05}}\underbrace{p_{50}}_{\frac{l_{51}}{l_{50}}} + v^2 \underbrace{p_{50}\,p_{51}}_{\frac{l_{52}}{l_{50}}}$$

$$= 1 + \frac{1}{1.05}\,\frac{773}{793} + \frac{1}{1.05^2}\,\frac{753}{750} = 2.789637$$

(1)

$$\ddot{a}^{(12)}_{50:\overline{3}|} = \underbrace{\ddot{a}^{(12)}_{50}}_{\alpha(m)\ddot{a}_{50} - \beta(m)} - {}_3E_{50}\,\underbrace{\ddot{a}^{(12)}_{53}}_{\alpha(m)\ddot{a}_{53} - \beta(m)}$$

$$= \alpha(m)\left[\ddot{a}_{50:\overline{3}|}\right] - \beta(m)\left[1 - {}_3E_{50}\right]$$

$${}_3E_{50} = v^3\frac{l_{53}}{l_{50}} = .7962992$$

$$\alpha(12) = \frac{i}{i^{(12)}}\,\frac{d}{d^{(12)}} = 1.000197$$

$$\beta(12) = \frac{i - i^{(m)}}{i^{(m)}\,d^{(m)}} = .466508$$

$$= \underline{2.69516}$$

(2)

$$\ddot{a}^{(m)}_x = \ddot{a}_x - \frac{m-1}{2m} \qquad \frac{11}{24}$$

$$\ddot{a}^{(12)}_{50:\overline{3}|} = \underbrace{\ddot{a}^{(12)}_{50}}_{-\frac{11}{24}} - {}_3E_{50}\,\underbrace{\ddot{a}^{(12)}_{53}}_{\frac{11}{24}} = \ddot{a}^{(12)}_{50:\overline{3}|} - \frac{11}{24}\left(1 - {}_3E_{50}\right) = \cancel{2.696276}$$

2.694516

③ $\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)$

W2 $\ddot{a}^{(12)}_{50:\overline{3}|} = \ddot{a}_{50:\overline{3}|} - \frac{11}{24}(1 - {}_3E_{50}) = 2.696276$

$\ddot{a}_{50:\overline{3}|}$: 2.789637

${}_3E_{50}$: .7961992

④ $\ddot{a}^{(12)}_{50:\overline{3}|} = \underbrace{\ddot{a}_{50:\overline{3}|} - \frac{12^2-1}{12(12^2)}\left[\delta + \mu_{50} - {}_3E_{50}(\delta + \mu_{53})\right]}_{= 2.695565}$

W3*

δ: 5%, μ_{50}: .0213, δ: 5%, μ_{53}: .0312

⑤ $\mu_{50} = -\frac{1}{2}\left[\log p_{49} - \log p_{50}\right] = .0239945$

W3* $\mu_{53} = -\frac{1}{2}\left[\log p_{52} - \log p_{53}\right] = .03151728$

$p_{50} = l_{51}/l_{50}$

$p_{49} = l_{50}/l_{49}$

$\vdots$

$\ddot{a}^{(12)}_{50:\overline{3}|} = 2.695143$

Practice problem 1

Sorry but this is discrete!

(In class I thought it was continuous)

So here comes the solution

correct

linear in x ⟹ De Moivre

You are given:

- $\ell_x = 115 - x$, for $0 \le x \le 115$
- $\delta = 4\%$

Calculate $\ddot{a}_{65:\overline{20}|}$.

Easier to work with insurance

$$A_{65:\overline{20}|} = \sum_{k=0}^{19} v^{k+1} \underbrace{{}_kp_{65}}_{\frac{\cancel{50-k}}{50}} \underbrace{q_{65+k}}_{\frac{1}{\cancel{50-k}}} + v^{20}\, {}_{20}p_{65}$$

$$= \frac{1}{50} \underbrace{\sum_{k=0}^{19} v^{k+1}}_{a_{\overline{20}|}} = \frac{1}{50}\left(\frac{1-v^{20}}{d}\right)$$

$d = 1 - v = 1 - e^{-\delta}$

$= 1 - e^{-.04}$

$v = e^{-.04}$

$$= \frac{e^{-.04}}{50}\left(\frac{1-e^{-.04(20)}}{1-e^{-.04}}\right) + e^{-.04(20)}\left(\frac{30}{50}\right)$$

$= 0.2698655 + 0.2695974$

$= 0.5394629$

Thus, $\ddot{a}_{65:\overline{20|}} = \dfrac{1 - A_{65:\overline{20|}}}{d}$

$$= \frac{1 - 0.5394629}{1 - e^{-.04}} = \underline{\underline{11.74523}}$$

verify this!

This is an example where using the CPT is much more difficult to do

$$\ddot{a}_{65:\overline{20|}} = \sum_{k=0}^{19} v^{k+1} {}_k p_{65} = \sum_{k=0}^{19} e^{-.04(k+1)} \left(1 - \frac{k}{50}\right)$$

$$= \sum_{k=0}^{19} e^{-.04(k+1)} - \frac{1}{50} \underbrace{\sum_{k=0}^{19} k e^{-.04(k+1)}}_{\text{This sum is more tricky to evaluate!}}$$

Practice problem 2

$Y = \bar{a}_{\overline{T|}}$

$E[Y] = \int_0^\infty v^t \, {}_tp_x \, dt = \frac{1}{.08}$

$e^{-.05t} \quad e^{-.03t} = e^{-.08t}$

$Var[Y] = \frac{1}{\delta^2}\left[{}^2\bar{A}_x - (\bar{A}_x)^2\right]$

$\bar{A}_x = \frac{\mu}{\mu+\delta} = \frac{3}{8} \qquad {}^2\bar{A}_x = \frac{\mu}{\mu+2\delta} = \frac{3}{13}$

$= \dfrac{\frac{3}{13} - \left(\frac{3}{8}\right)^2}{(.05)^2} = 36.05769$

You are given:

- $\mu_{x+t} = 0.03$, for $t \geq 0$
- $\delta = 5\%$
- Y is the present value random variable for a continuous whole life annuity of \$1 issued to (x).

Calculate $\Pr\left[Y \geq \mathrm{E}[Y] - \sqrt{\mathrm{Var}[Y]}\right]$.

$\frac{1}{.08} - \sqrt{36.05769} = 6.495194$

$$\Pr\Big[\underbrace{\bar{a}_{\overline{T}|}}_{\frac{1-v^T}{.05}} \geq 6.495194\Big] = \Pr\Big[\underbrace{v^T}_{e^{-\delta T}} \leq 1-.05(6.495194)\Big]$$

$$= \Pr\Big[T \geq \underbrace{-\frac{1}{.05}\log\big(1-.05(6.495194)\big)}_{a}\Big]$$

$$= \underbrace{\Pr[T \geq a]}_{e^{-.03a}}$$

if T is Exponential

$$f(t) = \mu e^{-\mu t}$$

$$F(t) = 1 - e^{-\mu t}$$

$$= e^{-.03a}$$

$$= 0.79$$

Practice problem 3 - modified SOA MLC Spring 2012

For a whole life annuity-due of \$1,000 per year on (65), you are given:

- Mortality follows Gompertz law with
$$\mu_x = Bc^x, \quad \text{for } x \geq 0,$$
where $B = 5 \times 10^{-5}$ and $c = 1.1$.
- $i = 4\%$
- Y is the present value random variable for this annuity.

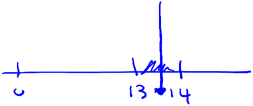
Calculate the probability that Y is less than \$11,500.

in terms of k $\iff$ $Y = 1000\,\ddot{a}_{\overline{K+1}|} =$

$$\Pr\left[\cancel{1000}\,\ddot{a}_{\overline{K+1}|} < \frac{11500}{1000} = 11.5\right] \qquad d = .04/1.04$$

$$\ddot{a}_{\overline{K+1}|} = \frac{1-v^{K+1}}{d} \quad \Longrightarrow \quad \Pr\left[v^{K+1} > 1 - 11.5d\right], \qquad v^{K+1} = e^{-\delta(K+1)}$$

$$\delta = \log(1.04) \quad \Longrightarrow \quad \Pr\Bigg[K < \underbrace{\frac{-1}{\log(1.04)}\log(1-11.5d) - 1}_{13.88876}\Bigg]$$



$$\underbrace{\Pr[K \le 13]}_{{}_{14}q_{65}} \quad \text{vs} \quad \cancel{\Pr[K < 14]}$$

$${}_{14}q_{65} = 1 - {}_{14}p_{65}$$

$$\mu_x = Bc^x \qquad \left.\begin{matrix} B \\ c \end{matrix}\right\} \text{given}$$

$${}_tp_x = e^{-\int_0^t Bc^{x+s}\,ds} = e^{-Bc^x \int_0^t c^s\,ds}, \qquad c^s = e^{s\log c}$$

$$= e^{-\frac{Bc^x}{\log c}(c^t - 1)}, \qquad x = 65,\ t = 14$$

$$= 1 - e^{-\frac{Bc^{65}}{\log(c)}(c^{14}-1)}$$

$$= 1 - e^{-.7196562} = \underline{.5130804}$$

Practice problem 4 - SOA MLC Spring 2014

For a group of 100 lives age x with independent future lifetimes, you are given:

- Each life is to be paid $1 at the beginning of each year, if alive.
- $A_x = 0.45$
- ${}^2A_x = 0.22$
- $i = 0.05$

$$Y = \sum_{i=1}^{100} Y_i$$

$$Y_i = \ddot{a}_{\overline{K+1}|}$$

$$E[Y_i] = \ddot{a}_x = \frac{1-A_x}{d} = \frac{1-.45}{.05/1.05}$$

11.55

$$Var[Y_i] = \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$$

Y is the present value random variable of the aggregate payments.

Using the Normal approximation to Y, calculate the initial size of the fund needed in order to be 95% certain of being able to make the payments for these life annuities.

$$7.7175 = \frac{1}{(.05/1.05)^2}\left[.22 - (.45)^2\right]$$

$E[Y] = 100(11.55) = 1155$

$Var[Y] = 100(7.7175) = 771.75$

F = fund needed

$Y = \Sigma$

$Y \overset{\text{approximately}}{\sim}$ normally dist

$N(1155, 771.75)$

$Pr[F \geq Y] = .95$ solve for F!

$$Pr\left[\frac{F-1155}{\sqrt{771.75}} \geq \underbrace{\frac{Y-E(Y)}{\sqrt{Var(Y)}}}_{Z \sim N(0,1)}\right] = Pr\left[Z \leq \underbrace{\frac{F-1155}{\sqrt{771.75}}}_{=1.645}\right] = .95$$

$1.645 \rightarrow 95^{th}$ percentile of $N(0,1)$

Solve for $F = \underline{\underline{1200.699}}$

greater probability $\Rightarrow$ need more funds!

Some special laws

Constant μ

$\bar{A}_x = \frac{\mu}{\mu+\delta}$

$\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = \frac{1-\frac{\mu}{\mu+\delta}}{\delta} = \frac{1}{\mu+\delta}$

$\int_0^\infty v^t \, {}_tp_x \, dt = \frac{1}{\mu+\delta}$

$v^t \to e^{-\delta t}$, ${}_tp_x \to e^{-\mu t}$

${}_{n|}\bar{a}_x = {}_nE_x \, \underbrace{\bar{a}_{x+n}}$

${}_nE_x \to e^{-\delta n} e^{-\mu n}$, $\bar{a}_{x+n} \to \frac{1}{\mu+\delta}$

$\frac{1-\bar{A}_{x:\overline{n}|}}{\delta}$

$\bar{a}_{x:\overline{n}|} = \bar{a}_x - {}_{n|}\bar{a}_x = \frac{1}{\mu+\delta}\left(1-e^{-(\mu+\delta)n}\right)$

~~ODE~~

De Moivre's

${}_tp_x \mu_{x+t} = \frac{1}{\omega-x}$

${}_tp_x =$

$\bar{A}_x = 1-\delta\bar{a}_x$

or

$\bar{a}_x = \frac{1-\bar{A}_x}{\delta}$

$\frac{1}{\omega-x}\bar{a}_{\overline{\omega-x}|}$

Other terminologies and notations used

Expression	Other terms/symbols used
temporary life annuity-due	term annuity-due n-year term life annuity-due
annuity-immediate	immediate annuity annuity immediate