Exercise 2.9

To verify the formula, we need the Leibnitz rule for differentiating an integral:

$$\frac{d}{dz}\int_{a(z)}^{b(z)} f(x,z)dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}dx + f(b(z),z)\frac{\partial b(z)}{\partial z} - f(a(z),z)\frac{\partial a(z)}{\partial z}$$

Therefore, we have

$$\begin{aligned} \frac{d}{dx}{}_tp_x &= \frac{d}{dx}\exp\left(-\int_x^{x+t}\mu_s ds\right) \\ &= -\exp\left(-\int_x^{x+t}\mu_s ds\right)\cdot\frac{d}{dx}\int_x^{x+t}\mu_s ds \\ &= -{}_tp_x(\mu_{x+t}-\mu_x) \\ &= {}_tp_x(\mu_x-\mu_{x+t}), \end{aligned}$$

where we applied the Leibnitz rule in the second step above.

Generally, because the force of mortality μ_x increases with age, we would expect $\frac{d}{dx}{}_tp_x$ to be non-positive. This implies that as we grow older with age, the rate of change of surviving for another fixed t years decreases.