Exercise 3.9

(a) Let the constant force between ages $[x+k, x+k+1]$ be denoted by μ^*_{x+k} so that

$$p_{x+k} = e^{-\mu^*_{x+k}},$$

from which it follows that $\mu^*_{x+k} = -\log p_{x+k}$. Therefore, we have

$$\begin{aligned}
\Pr[R_x \le s | K_x = k] &= \frac{\Pr[R_x \le s, K_x = k]}{\Pr[K_x = k]} = \frac{\Pr[k < T_x \le k+s]}{\Pr[K_x = k]} \\
&= \frac{{}_kp_x \cdot {}_sq_{x+k}}{{}_kp_x \cdot q_{x+k}} = \frac{{}_sq_{x+k}}{q_{x+k}} = \frac{1 - {}_sp_{x+k}}{1 - p_{x+k}} \\
&= \frac{1 - \exp\left\{ -\int_0^s \mu^*_{x+k} dz \right\}}{1 - \exp\{-\mu^*_{x+k}\}} = \frac{1 - \exp\{-\mu^*_{x+k} s\}}{1 - \exp\{-\mu^*_{x+k}\}}.
\end{aligned}$$

It is not difficult to see that we can also write this as

$$\Pr[R_x \le s | K_x = k] = \frac{1 - (p_{x+k})^s}{1 - p_{x+k}}.$$

(b) For $0 \le s \le 1$, we note that

$$\begin{aligned}
\mu_{x+k+s} &= -\frac{d}{ds} \log {}_sp_{x+k} = -\frac{d}{ds} \log (p_{x+k})^s \\
&= -\frac{d}{ds} s \cdot \log p_{x+k} = -\log p_{x+k} = \mu^*_{x+k},
\end{aligned}$$

and is therefore constant between any integral ages $x+k$ and $x+k+1$. The result therefore follows.