Exercise 4.5

(a) Endowment insurance is the sum of a term insurance and a pure endowment, that is, $A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;\;\;1}$. Therefore, we have

$$\begin{aligned}
A_{x:\overline{n}|} &= \sum_{k=0}^{n-1} v^{k+1}\,{}_{k|}q_x + v^n\,{}_np_x \\
&= \sum_{k=0}^{n-2} v^{k+1}\,{}_{k|}q_x + v^n\,{}_{n-1|}q_x + v^n\,{}_np_x \\
&= \sum_{k=0}^{n-2} v^{k+1}\,{}_{k|}q_x + v^n\left[{}_{n-1}p_x\,(1-p_{x+n-1}) + {}_np_x\right] \\
&= \sum_{k=0}^{n-2} v^{k+1}\,{}_{k|}q_x + v^n[{}_{n-1}p_x - \underbrace{{}_{n-1}p_x\,p_{x+n-1}}_{{}_np_x} + {}_np_x] \\
&= \sum_{k=0}^{n-2} v^{k+1}\,{}_{k|}q_x + v^n\,{}_{n-1}p_x,
\end{aligned}$$

which proves the result.

(b) The primary difference lies in the payment after period $n-1$. If (x) survives to live for $n-1$ years, he will receive a benefit at the end of year n if he dies the following year, and if he survives, he will receive the pure endowment. So once he reaches age $x+n-1$, a benefit of \$1 is payable at the end of year n, regardless of whether he survives or not. Hence, this explains the second term $v^n\,{}_{n-1}p_x$ in the equation in part (a).