Exercise 5.11

(a) Let $Y = a_{\overline{\min(K,n)}|}$. As an aside, note that Y is the present value random variable associated with an n-year life annuity-immediate on (x). Rewriting Y as

$$Y = \frac{1 - v^{\min(K,n)}}{i} \cdot \frac{v}{v} = \frac{v - v^{\min(K+1,n+1)}}{iv} = \frac{v - v^{\min(K+1,n+1)}}{d},$$

we find

$$\mathrm{Var}[Y] = \frac{1}{d^2}\mathrm{Var}\left[v^{\min(K+1,n+1)}\right] = \frac{{}^2A_{x:\overline{n+1}|} - \left(A_{x:\overline{n+1}|}\right)^2}{d^2}$$

because $v^{\min(K+1,n+1)}$ is the present value random variable of an $(n+1)$-year endowment to (x).

(b) Now, express Y as

$$Y = \frac{1}{i}\left[1 - \left(v^K I(K < n) + v^n I(K \geq n)\right)\right]$$

and define $Z_1 = v^K I(K < n)$ and $Z_2 = v^n I(K \geq n)$ so that clearly $Y = \frac{1}{i}\left[1 - (Z_1 + Z_2)\right]$. The variance of Y therefore can be written as

$$\mathrm{Var}[Y] = \frac{1}{i^2}\mathrm{Var}[Z_1 + Z_2] = \frac{1}{i^2}\left\{\mathrm{Var}[Z_1] + 2\mathrm{Cov}[Z_1, Z_2] + \mathrm{Var}[Z_2]\right\}. \quad (1)$$

We note that

$$\mathrm{Var}[Z_1] = \frac{1}{v^2}\mathrm{Var}\left[v^{K+1} I(K < n)\right] = (1+i)^2\left[{}^2A^{1}_{x:\overline{n}|} - \left(A^{1}_{x:\overline{n}|}\right)^2\right]. \quad (2)$$

Since Z_2 is the present value random variable of an n-year pure endowment and that $Z_1 Z_2 = 0$, we find

$$\mathrm{Cov}[Z_1, Z_2] = -\mathrm{E}[Z_1] \cdot \mathrm{E}[Z_2] = -\frac{1}{v}A^{1}_{x:\overline{n}|} \cdot v^n\,{}_np_x = -(1+i)A^{1}_{x:\overline{n}|} \cdot v^n\,{}_np_x \quad (3)$$

and that

$$\mathrm{Var}[Z_2] = v^{2n}\,{}_np_x(1 - {}_np_x). \quad (4)$$

Finally, plugging the results of (2), (3) and (4) into (1), we get the desired result.